Review of Matrix Algebra

Matrices

- A *matrix* is a rectangular or square array of values arranged in rows and columns.
- An $m \times n$ matrix $\mathbf{A}$, has m rows and n columns, and has a general form of

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Examples of Matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & 2 \end{bmatrix}$$ is a 2 × 3 matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 6 \\ 7 & 2 \\ 2 & 9 \end{bmatrix}$$ is a 4 × 2 matrix

Vectors

- A *vector* is a matrix with only one column (a *column vector*) or only one row (a *row vector*).

Examples of Vectors

$$\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$$ is a column vector.

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 & 1 & 5 \end{bmatrix}$$ and $$\mathbf{y}' = \begin{bmatrix} 4 & 7 & -5 \end{bmatrix}$$

are row vectors.

Scalars

- A single number such as 2.4 or –6 is called a *scalar*.
- The elements of a matrix are usually scalars, although a matrix can be expressed as a matrix of smaller matrices.

Vector Addition

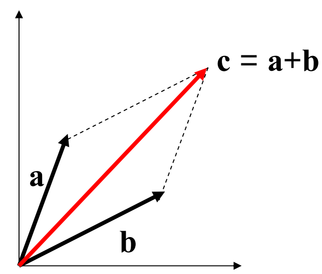
Example:

Given: vector $\mathbf{a}$ = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

vector $\mathbf{b}$ = $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Then: vector $\mathbf{c}$ = $\mathbf{a} + \mathbf{b}$

= $\begin{bmatrix} 1+3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



Multiplication by a Scalar

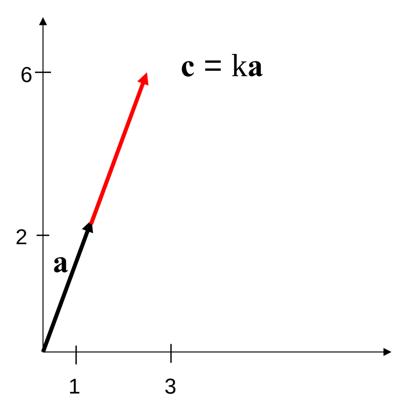
Example:

Given: vector $\mathbf{a}$ = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

scalar k = 3

Then: vector $\mathbf{c} = k\mathbf{a}$

= $\begin{bmatrix} 3*1 \\ 3*2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



Transpose

- The *transpose* of a matrix $\mathbf{A}$ is the matrix whose columns are rows of $\mathbf{A}$ (and therefore whose rows are columns of $\mathbf{A}$), with order retained, from first to last.
- The transpose of $\mathbf{A}$ is denoted by $\mathbf{A}'$.

Example of Matrix Transpose

Let $\mathbf{A} = \begin{bmatrix} 3 & 2 & -6 \\ 7 & 1 & 2 \end{bmatrix}$

Then $\mathbf{A}' = \begin{bmatrix} 3 & 7 \\ 2 & 1 \\ -6 & 2 \end{bmatrix}$

Matrix Transpose

- If $\mathbf{A}$ is 2×3, then $\mathbf{A}'$ is 3×2.
- In general if $\mathbf{A}$ is $m \times n$, then $\mathbf{A}'$ is $n \times m$, and $a'_{ij} = a_{ji}$
- The transpose of a row vector is a column vector

Partitioned Matrices

- The matrix $\mathbf{A}$ can be written as a matrix of matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

- This specification of $\mathbf{A}$ is called a *partitioning* of $\mathbf{A}$, and the matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$ are said to be submatrices of $\mathbf{A}$.
- $\mathbf{A}$ is called a *partitioned matrix*.

Example of Partitioned Matrix

$$\mathbf{A} = \left[\begin{array}{cccc:cc} 1 & 6 & 9 & 4 & 5 & 8 \\ 5 & 4 & 7 & 2 & 0 & 2 \\ 9 & 2 & 8 & 1 & 7 & 1 \\ \hdashline 9 & 1 & 7 & 6 & 2 & 3 \\ 2 & 5 & 4 & 8 & 1 & 7 \end{array}\right]$$

Each of the arrays of numbers in the four sections of $\mathbf{A}$ engendered by the dashed lines is a matrix.

Example of Partitioned Matrix

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 6 & 9 & 4 \\ 5 & 4 & 7 & 2 \\ 9 & 2 & 8 & 1 \end{bmatrix} \qquad \mathbf{A}_{12} = \begin{bmatrix} 5 & 8 \\ 0 & 2 \\ 7 & 1 \end{bmatrix}$$

$$\mathbf{A}_{21} = \begin{bmatrix} 9 & 1 & 7 & 6 \\ 2 & 5 & 4 & 8 \end{bmatrix} \qquad \mathbf{A}_{22} = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

Trace

- The sum of the diagonal elements of a square matrix is called the *trace* of the matrix, that is,

$$\begin{aligned}\text{tr}(\mathbf{A}) &= a_{11} + a_{22} + \cdots + a_{nn} \\ &= \sum_{i=1}^{n} a_{ii}\end{aligned}$$

Example of Trace

Let $\mathbf{A} = \begin{bmatrix} 1 & 5 & 9 \\ -3 & 2 & 8 \\ 4 & 7 & 6 \end{bmatrix}$

Then $\text{tr}(\mathbf{A}) = 1 + 2 + 6 = 9$

Addition and Subtraction

- Matrices of the <u>same size</u> are added or subtracted by adding or subtracting corresponding elements.

Example of Matrix Addition

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 \\ 9 & -1 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & -7 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2+1 & 4+5 & 8+3 \\ 9+2 & (-1)+3 & 5+(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 9 & 11 \\ 11 & 2 & -2 \end{bmatrix}$$

Example of Matrix Subtraction

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 \\ 9 & -1 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & -7 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2-1 & 4-5 & 8-3 \\ 9-2 & -1-3 & 5-(-7) \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -1 & 5 \\ 7 & -4 & 12 \end{bmatrix}$$

Multiplication: Inner Product

The *inner product* of two vectors

$$\mathbf{a}' = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is defined as

$$\mathbf{a}'\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i$$

Example of Inner Product Multiplication

Let $\mathbf{a}' = \begin{bmatrix} 3 & 1 & 10 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 10 \\ 3 \end{bmatrix}$

then the inner product

$\mathbf{a}'\mathbf{x}$ = 3(5) + 1(10) + 10(3) = 55.

Matrix Multiplication

- Two matrices $\mathbf{A}$ and $\mathbf{B}$ is are conformable to matrix multiplication only if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$.
- $\mathbf{AB}$ has the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$.
- The ij^{th} element of $\mathbf{AB}$ is the inner product of the i^{th} row of $\mathbf{A}$ and j^{th} column of $\mathbf{B}$.

Matrix Multiplication

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

Example of Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 6 & 1 & 5 \\ 2 & 1 & -2 & 3 \\ 4 & 1 & 2 & 5 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1(0)+2(2)+3(4) & 1(6)+2(1)+3(1) & 1(1)+2(-2)+3(2) & 1(5)+2(3)+3(5) \\ -1(0)+4(2)+2(4) & -1(6)+4(1)+2(1) & -1(1)+4(-2)+2(2) & -1(5)+4(3)+2(5) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 11 & 3 & 26 \\ 16 & 0 & -5 & 17 \end{bmatrix}$$

Matrix Multiplication

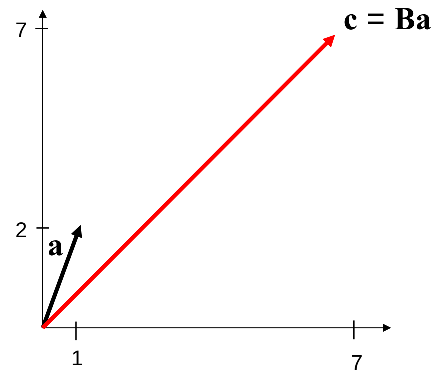
Example:

Given: vector $\mathbf{a}$ = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

matrix $\mathbf{B}$ = $\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$

Then: vector $\mathbf{c}$ = $\mathbf{Ba}$

= $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$



Properties of Matrix Multiplication

1. $\mathbf{AB} \neq \mathbf{BA}$ — noncommutative
2. $\mathbf{(AB)C} = \mathbf{A(BC)}$ — associative
3. $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$ — left distributive
   $\mathbf{(B+C)D} = \mathbf{BD} + \mathbf{CD}$ — right distributive
4. $k\,\mathbf{(AB)} = (k\,\mathbf{A})\mathbf{B}$ — where k is a scalar
5. $\mathbf{(AB)}' = \mathbf{B}'\mathbf{A}'$

Direct or Kronecker Product $\otimes$

- Suppose $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $p \times q$.
- Then the *direct* or *Kronecker product* $\mathbf{A}\otimes\mathbf{B}$ is of size $mp \times nq$ and is most easily described as the partitioned matrix:

Direct or Kronecker Product $\otimes$

$$\mathbf{A}_{m\times n} \otimes \mathbf{B}_{p\times q} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \dots & \dots & \dots & \dots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

Example of Kronecker Product

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & 2 & 0 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \left[\begin{array}{cccc:cccc} 1 & -1 & 1 & 2 & 2 & -2 & 2 & 4 \\ 3 & 2 & 0 & 1 & 6 & 4 & 0 & 2 \\ -1 & 0 & 2 & 3 & -2 & 0 & 4 & 6 \\ \hdashline -1 & 1 & -1 & -2 & 3 & -3 & 3 & 6 \\ -3 & -2 & 0 & -1 & 9 & 6 & 0 & 3 \\ 1 & 0 & -2 & -3 & -3 & 0 & 6 & 9 \end{array}\right]$$

Square Matrix

- A *square* matrix is a matrix whose number of columns equals the number of rows.
- The elements on the diagonal,
  $a_{11}, a_{22}, \ldots, a_{nn}$,
  are referred to as the *diagonal elements* or *diagonal* of the matrix.
- Elements of a square matrix other than the diagonal elements are called *off-diagonal* or *nondiagonal* elements.

Diagonal Matrix

- A *diagonal* matrix is a square matrix having all its off-diagonal elements equal to zero.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Triangular Matrix

- A *triangular* matrix is a square matrix with all elements above (or below) the diagonal being equal to zero.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 \\ 0 & -2 & 9 \\ 0 & 0 & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ -7 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

A is an *upper triangular matrix* and
B is a *lower triangular matrix*

Identity Matrix

- A diagonal matrix having all diagonal elements equal to one is called an *identity* matrix.
- It is denoted by the letter $\mathbf{I}$.
- For any matrix $\mathbf{A}$ of order $m \times n$,

$$\mathbf{I}_m\mathbf{A}_{m\times n} = \mathbf{A}_{m\times n}\mathbf{I}_n = \mathbf{A}_{m\times n}$$

Example of Identity Matrix

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix of order 4.

Symmetric Matrix

- A *symmetric* matrix is a square matrix that is equal to its transpose.

Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Then $\mathbf{A}' = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \mathbf{A}$

In symmetric matrices,
the area above the diagonal is
a mirror image
of the area below the diagonal.

Summing Vectors

- Vectors whose every element is unity are called *summing vectors* because they can be used to express a sum of numbers in matrix notation as an inner product.

Example of Summing Vector

$\mathbf{1}' = [1 \quad 1 \quad 1 \quad 1]$ is a summing vector of order 4.

Let $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 8 \end{bmatrix}$

Then $\mathbf{1}'\mathbf{x} = [1 \quad 1 \quad 1 \quad 1] \begin{bmatrix} 2 \\ 4 \\ -3 \\ 8 \end{bmatrix} = 2 + 4 - 3 + 8 = \mathbf{x}'\mathbf{1}$

More on Summing Vectors

$\mathbf{1}_n'\mathbf{1}_n = n$

$\mathbf{1}_m\mathbf{1}_n' = \mathbf{J}_{m \times n}$ a **J** matrix having all elements equal to one.

$\mathbf{1}_n\mathbf{1}_n' = \mathbf{J}_n$ a square **J** matrix.

The $\mathbf{J}$ Matrix

$$\mathbf{J}_n = \mathbf{1}_n\mathbf{1}'_n$$

$$\mathbf{J}_n^2 = \left(\mathbf{1}_n\mathbf{1}'_n\right)\left(\mathbf{1}_n\mathbf{1}'_n\right) = \mathbf{1}_n\left(\mathbf{1}'_n\mathbf{1}_n\right)\mathbf{1}'_n = n\mathbf{J}_n$$

Idempotent Matrix

An idempotent matrix $\mathbf{K}$ satisfies $\mathbf{K}^2 = \mathbf{K}$.

Let $\overline{\mathbf{J}}_n = \frac{1}{n}\mathbf{J}_n$

Then $\overline{\mathbf{J}}_n^2 = \overline{\mathbf{J}}_n$

Centering Matrix

$$\mathbf{C}_n = \mathbf{I} - \bar{\mathbf{J}}_n = \mathbf{I} - \frac{1}{n}\mathbf{J}_n$$

is known as a centering matrix.

$$\mathbf{C} = \mathbf{C}' = \mathbf{C}^2$$

$$\mathbf{C1} = \mathbf{0} \qquad \mathbf{CJ} = \mathbf{JC} = \mathbf{0}$$

Orthogonal Matrix

- An *orthogonal* matrix $\mathbf{A}$ is a matrix having the property

$$\mathbf{AA}' = \mathbf{I} = \mathbf{A}'\mathbf{A}$$

Example of Orthogonal Matrix

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{bmatrix}$$

Helmert Matrix

- A Helmert matrix is a type of an orthogonal contrast matrix.
  - The first row constructs an average
  - The second row contrasts the second element with the first element
  - The third row contrasts the third element with the average of the first two elements
  - The fourth row contrasts the fourth element with the average of the first three elements
  - And so forth

Example of Helmert Matrix

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} & 1/\sqrt{4} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & -3/\sqrt{12} \end{bmatrix}$$

Quadratic Form

- A *quadratic form* is the product of a row vector $\mathbf{x}'$, a matrix $\mathbf{A}$, and the column vector $\mathbf{x}$, that is, $\mathbf{x}'\mathbf{A}\mathbf{x}$.
- This is a quadratic function of the x's. Notice that to result in the same quadratic function of x's, you can use many different matrices.
- Each matrix has the same diagonal elements, and the sum of each pair of symmetrically placed off-diagonal elements a_{ij} and a_{ji} is the same.

Quadratic Form

- For any particular quadratic form there is a unique *symmetric matrix* $\mathbf{A}$ for which the quadratic form can be expressed as $\mathbf{x}'\mathbf{A}\mathbf{x}$:

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i^2 a_{ii} + 2\sum_{j=i+1}^{n}\sum_{i=1}^{n} x_i x_j a_{ij}$$

Example of a Quadratic Form

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\mathbf{x}$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3)$$

Example of a Quadratic Form

When $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 4 \end{bmatrix}$

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 5x_2^2 + x_3^2 + 4x_1x_2 + 6x_2x_3 + 2x_2x_3$$

Positive Definite Quadratic Form and Positive Definite Matrix

- When $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x}$ other than $\mathbf{x} = \mathbf{0}$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is a *positive definite* quadratic form;
- $\mathbf{A} = \mathbf{A}'$ is correspondingly a *positive definite matrix*.

Example of Positive Definite Quadratic Form and Matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The matrix A is positive definite

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_2x_3 + 2x_2x_3$$

$$= (x_1 + 2x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2 > 0$$

other than $\mathbf{x} = 0$

Positive Semidefinite Quadratic Form and Positive Semidefinite Matrix

- When $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$ and $\mathbf{x}'\mathbf{A}\mathbf{x} = 0$ for some $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}'\mathbf{A}\mathbf{x}$ is a *positive semidefinite* quadratic form;
- $\mathbf{A} = \mathbf{A}'$ is correspondingly a *positive semidefinite matrix*.

Nonnegative Definite Quadratic Form and Nonnegative Definite Matrix

- The two classes of matrices taken together, positive definite and positive semidefinite, are called *nonnegative definite* matrices.

Example of Positive Definite Quadratic Form and Matrix

Let $\mathbf{A} = \begin{bmatrix} 37 & -2 & -24 \\ -2 & 13 & -3 \\ -24 & -3 & 17 \end{bmatrix}$

Then $\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 37 & -2 & -24 \\ -2 & 13 & -3 \\ -24 & -3 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= 37x_1^2 + 13x_2^2 + 17x_3^2 - 4x_1x_2 - 48x_2x_3 - 6x_2x_3$$

$$= (x_1 - 2x_2)^2 + (6x_1 - 4x_3)^2 + (3x_2 - x_3)^2$$

This is zero for $\mathbf{x}'$ = [2 1 3] and for any scalar multiple thereof, as well as for $\mathbf{x}$ = **0**.

Example of Positive Definite Quadratic Form and Matrix

$$\mathbf{x}'\mathbf{C}\mathbf{x} = \mathbf{x}'\left(\mathbf{I} - \overline{\mathbf{J}}\right)\mathbf{x} = \sum_{i=1}^{n}(x_i - \overline{x})^2$$

is a positive semidefinite quadratic form because it is positive, except for being zero when all the x_is are equal. Its matrix, $\mathbf{I} - \overline{\mathbf{J}}$ which is idempotent, is also positive semidefinite, as are all symmetric idempotent matrices (except $\mathbf{I}$, which is the only positive definite idempotent matrix).

Determinant of a Square Matrix

- The *determinant* of a square matrix of order n,
  (that is, $\mathbf{A}_{n\times n} = (a_{ij})$; $i, j = 1, 2, \ldots, n$)
  is the sum of all possible products of n elements of $\mathbf{A}$ such that
  1. each product has one and only one element from every row and column of $\mathbf{A}$,

Determinant of a Square Matrix

2. the sign of a product being $(-1)^p$ for

   $p = \sum_{i=1}^{n} n_i$ where by writing

   a. the product with its i subscripts in natural order

      $a_{1j_1} a_{2j_2} \cdots a_{ij_i} \cdots a_{nj_n}$

   b. The j subscripts j_i, i = 1, 2, …, n, being the first n integers in some order.

   n_i is defined as the number of j's less than j_i that follow j_i in this order.

Determinant of a Square Matrix

- The determinant of a square matrix $\mathbf{A}$, denoted by $|\mathbf{A}|$, is a polynomial of the elements of a square matrix.
- It is a scalar.
- It is the sum of certain products of the elements of the matrix from which it is derived, each product being multiplied by +1 or –1 according to certain rules.

Example:
Determinant of a 2 × 2 Matrix

$$|\mathbf{A}| = \begin{vmatrix} 3 & 7 \\ 2 & 6 \end{vmatrix} = 3(6) - 7(2)$$

Example:
Determinant of a 3 × 3 Matrix

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix}$$

$$= 1(+1)\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} + 2(-1)\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} + 3(+1)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(50 - 48) - 2(40 - 42) + 3(32 - 35)$$

$$= -3$$

Example: Determinant of a 3 × 3 Matrix

- The determinant that multiplies each element of the chosen row (in this case, the first row) is the determinant derived from $|\mathbf{A}|$ by crossing out the row and column containing the element concerned.
- For example, the first element, 1, is multiplied by the determinant

$$\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix}$$

which is obtained from $|\mathbf{A}|$ through crossing out the first row and column.

Minors

- Determinants obtained in this way are called *minors* of $|\mathbf{A}|$, that is,

$$\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix}$$

is the minor of the element 1 in $|\mathbf{A}|$, and

$$\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix}$$

is the minor of the element 2 in $|\mathbf{A}|$.

Determinants and Minors

- The (+1) and (–1) factors in the expansion are decided on according to the following rule:
  - If $\mathbf{A}$ is written in the form $\mathbf{A} = (a_{ij})$, the product of a_{ij} and its minor in the expansion of determinant $|\mathbf{A}|$ is multiplied by $(-1)^{i+j}$.
- Therefore, because the element 1 in the example is the element a_{11}, its product with its minor is multiplied by $(-1)^{1+1} = +1$. For element 2, which is a_{12}, its product with its minor is multiplied by $(-1)^{1+2} = -1$.

Expansion by Minors

- Denote the minor of the element a_{ij} by $|\mathbf{M}_{ij}|$, where $\mathbf{M}_{ij}$ is a submatrix of $\mathbf{A}$ obtained by deleting the i^{th} row and the j^{th} column.
- The determinant of an *n*-order matrix is obtained by *expansion by the elements of a row* (*or column*), or *expansion by minors*:

Expansion by Minors

$$\left|\mathbf{A}\right| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \, |\,\mathbf{M}_{ij}\,| \quad \text{for any row } i\text{, or}$$

$$\left|\mathbf{A}\right| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \, |\,\mathbf{M}_{ij}\,| \quad \text{for any column } j.$$

Cofactors

- The signed minor $(-1)^{i+j} |\mathbf{M}_{ij}|$ is called a *cofactor*:

  $c_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$.

Matrix Inverses

- The *inverse* of a square matrix $\mathbf{A}$ is a matrix whose product with $\mathbf{A}$ is the identity matrix $\mathbf{I}$.
- The inverse matrix is denoted by $\mathbf{A}^{-1}$.
- The concept of "dividing" by $\mathbf{A}$ in matrix algebra is replaced by the concept of multiplying by the inverse matrix $\mathbf{A}^{-1}$.

Matrix Inverses

- An inverse matrix $\mathbf{A}^{-1}$ has the following properties:
  - $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$
  - $\mathbf{A}^{-1}$ unique for given $\mathbf{A}$.

Adjugate or Adjoint

- An *adjugate* (or *adjoint*) of matrix $\mathbf{A}$, denoted by adj($\mathbf{A}$), is obtained by replacing the elements in $\mathbf{A}$ by their cofactors and then transposing it.

$$\mathbf{A}^{-1} = \frac{1}{|\,\mathbf{A}\,|}\,\text{adj}\,(\mathbf{A})$$

Example of Matrix Inverse

Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$

The determinant of $\mathbf{A}$ is:

$$|\mathbf{A}| = \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} = 18 - 15 = 3$$

Example of Matrix Inverse

Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$

- The cofactor for a_{11} = 2 is $(-1)^{1+1}$ |9| = 9.
- The cofactor for a_{12} = 5 is $(-1)^{1+2}$ |3| = –3.
- The cofactor for a_{21} = 3 is $(-1)^{2+1}$ |5| = –5.
- The cofactor for a_{22} = 9 is $(-1)^{2+2}$ |2| = 2.

Example of Matrix Inverse

The adjoint matrix is $\begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix}$

So the inverse is: $\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix}$

Conditions for Existence of a Matrix Inverse

1. $\mathbf{A}^{-1}$ can exist only when $\mathbf{A}$ is square.
2. $\mathbf{A}^{-1}$ can exist only if $|\mathbf{A}|$ is nonzero.

Properties of Matrix Inverses

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$$ k a scalar

$$\mathbf{D}^{-1} = diag\left(\mathrm{d}_{11}^{-1}, \cdots, \mathrm{d}_{nn}^{-1}\right)$$ $\mathbf{D}$ a diagonal matrix

Singular Matrix

- Let the square matrix $\mathbf{A}_{n\times n}$ be of order n.
- The matrix $\mathbf{A}$ is said to be singular if any of the following equivalent conditions exists:
  i. $|\mathbf{A}| = 0$
  ii. If a particular row (column) can be formed as a linear combination of the other rows (columns)
  iii. $r(\mathbf{A}) < n$

Linear Combination of Vectors

Let $\mathbf{X}$ be the matrix having columns $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$, and let $\mathbf{a}$ be the vector of *a*'s:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

Linear Combination of Vectors

Then the *linear combination* of the set of *n* vectors is:

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = \sum_{i=1}^{n} a_i\mathbf{x}_i = \mathbf{Xa}$$

Linear Combination of Vectors

- $\mathbf{Xa}$ is a column vector, a linear combination of the columns of $\mathbf{X}$.
- $\mathbf{b'X}$ is a row vector, a linear combination of the rows of $\mathbf{X}$.
- $\mathbf{AB}$ is a matrix. Its rows are linear combinations of the rows of $\mathbf{B}$, and its columns are linear combinations of the columns of $\mathbf{A}$.

Example of Linear Combination

$$\mathbf{X} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 1 \\ -1 & 3 & 5 \\ 6 & 7 & 5 \end{bmatrix} \qquad \mathbf{a}' = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

$$\mathbf{Xa} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 1 \\ -1 & 3 & 5 \\ 6 & 7 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 - 2a_2 + 0a_3 \\ 0a_1 + 4a_2 + a_3 \\ -a_1 + 3a_2 + 5a_3 \\ 6a_1 + 7a_2 + 5a_3 \end{bmatrix}$$

Linearly Independent Vectors

If there exists a vector $\mathbf{a} \neq \mathbf{0}$, such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = \mathbf{0}$$

then provided none of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is null, those vectors (the columns of **X**) are said to be *linearly independent vectors*.

Rank of a Matrix

- The *rank* of a matrix is the number of linearly independent rows (and columns) in the matrix.
- The rank of $\mathbf{A}$ is denoted by $r(\mathbf{A})$.

Rank of a Matrix

- When $r(\mathbf{A}_{n\times n}) = n$, then $\mathbf{A}$ is *nonsingular*, that is, $\mathbf{A}^{-1}$ exists.
- When $r(\mathbf{A}_{p\times q}) = p < q$, then $\mathbf{A}$ has *full row rank*, that is, its rank equals its number of rows.
- When $r(\mathbf{A}_{p\times q}) = q < p$, then $\mathbf{A}$ has *full column rank*, that is, its rank equals its number of columns.

Rank of a Matrix

- When $r(\mathbf{A}_{n\times n}) = n$,
  - $\mathbf{A}$ has *full rank*, that is, its rank equals its order,
  - it is nonsingular,
  - its inverse exists,
  - and it is called *invertible*.

Example of Rank and Linear Dependence

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix} \quad \mathbf{x}_5 = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$2\mathbf{x}_1 + \mathbf{x}_4 = \begin{bmatrix} 6 \\ -12 \\ 18 \end{bmatrix} + \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix} = \mathbf{0}$$

$\mathbf{x}_1$ and $\mathbf{x}_4$ are linearly dependent Vectors.

Example of Rank and Linear Dependence

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix} \quad \mathbf{x}_5 = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$2\mathbf{x}_1 + 3\mathbf{x}_2 - 3\mathbf{x}_3 = \begin{bmatrix} 6 \\ -12 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ 15 \\ -15 \end{bmatrix} + \begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, are linearly dependent vectors.

Example of Rank and Linear Dependence

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} -6 \\ 12 \\ -18 \end{bmatrix} \quad \mathbf{x}_5 = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = \begin{bmatrix} 3a_1 \\ -6a_1 \\ 9a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5a_2 \\ -5a_2 \end{bmatrix} = \begin{bmatrix} 3a_1 \\ -6a_1 + 5a_2 \\ 9a_1 - 5a_2 \end{bmatrix}$$

There are no values $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which makes it a null vector other than $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \mathbf{0}$. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ are *linearly independent* vectors.

Example of Rank and Linear Dependence

The rank of the matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 & -6 & 2 \\ -6 & 5 & 1 & 12 & -3 \\ 9 & -5 & 1 & -18 & 3 \end{bmatrix}$$

is 3

Rank of a Matrix

Any nonnull matrix **A** of rank *r* is equivalent to

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{C}$$

where $\mathbf{I}_r$ is the identity matrix of order *r*,
and the null submatrices are of appropriate order
to make $\mathbf{C}$ the same order as $\mathbf{A}$.

Rank of a Matrix

Any nonnull matrix $\mathbf{A}$ of rank *r* is equivalent to

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{C}$$

For $\mathbf{A}$ of order $m \times n$, $\mathbf{P}$ and $\mathbf{Q}$ are nonsingular matrices of order *m* and *n*, respectively, being products of elementary operators.

The matrix $\mathbf{C}$ is called the *equivalent canonical form*. It always exists, and can be used to determine the rank of $\mathbf{A}$.

Eigenvalues and Eigenvectors

Eigenvalue (also known as a characteristic root):

- the consolidated variance of a square matrix
- the variance accounted for by a linear combination of the input variables.

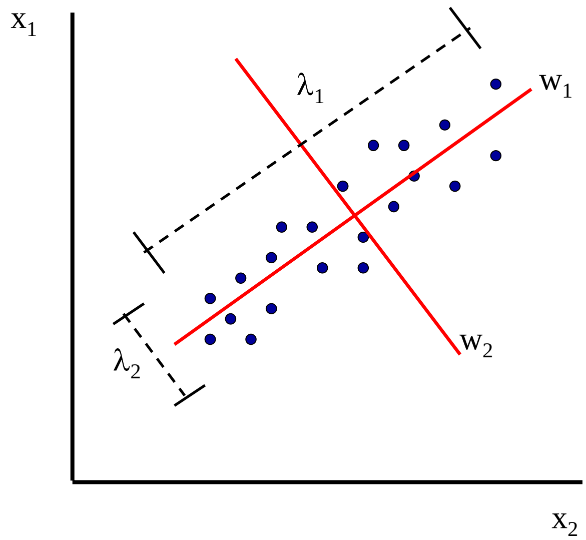
Eigenvalues can have values of zero.

Eigenvector (also known as a characteristic vector):

- a nonzero vector that is a linear combination of a set of variables maximizing shared variance among p variables

If a matrix $\mathbf{X}$ is of size $p \times p$, there are p eigenvalues.

Eigenvalues and Eigenvectors



Eigenvalues & Eigenvectors

Consider equations

$$\mathbf{Ax} = \lambda\mathbf{x} \;\; \textit{or} \;\; (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

for $\mathbf{x}$ (a vector) and λ (a scalar), and solve for ***x.***

If the matrix $\mathbf{A}-\lambda\mathbf{I}$ is nonsingular, the unique solution to these equations is $\mathbf{x} = \mathbf{0}$. You only get a nontrivial solution when $|\mathbf{A}-\lambda\mathbf{I}| = 0$.

Eigenvalues

If you expand this determinant $|\mathbf{A}-\lambda\mathbf{I}|$, it becomes a polynomial equation in λ of degree *p*.

This is called the *characteristic equation* of **A***.* Its *p* roots (which may be real or complex, simple or multiple) are called *eigenvalues* (or proper values, characteristic values, or latent roots) of **A***.*

The eigenvalues of A are denoted in order by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Eigenvectors

If λ is an eigenvalue, a nonzero vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$ is called an eigenvector (or proper vector, characteristic vector, or latent vector) corresponding to λ.

It is often convenient to normalize each eigenvector to have a squared length of 1.

Example of Eigenvalues

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$

has a characteristic equation

$$\left| \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)^2 - 36 = 0$, or $\lambda = -5$ or 7

Example of Eigenvectors

It can be seen that

$$\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = -5\begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue –5

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 7

Properties of Eigenvalues & Eigenvectors

For $\mathbf{Ax} = \lambda\mathbf{x}$,

1. $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$ and $\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$, when $\mathbf{A}$ is nonsingular.
2. $c\mathbf{Ax} = c\lambda\mathbf{x}$ for any scalar c.
3. $f(\mathbf{A})\mathbf{x} = f(\lambda)\mathbf{x}$ for any polynomial function $f(\mathbf{A})$

Properties of Eigenvalues & Eigenvectors

For $\mathbf{Ax} = \lambda\mathbf{x}$,

4. $\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(\mathbf{A})$ $\qquad$ $\prod_{i=1}^{n} \lambda_i = |\mathbf{A}|$

the sum of eigenvalues of a matrix equals its trace,

and their product equals its determinant.

Properties of Eigenvalues & Eigenvectors

For $\mathbf{Ax} = \lambda\mathbf{x}$,

5. If $\mathbf{A}$ is symmetric, then
   - the eigenvalues of matrix $\mathbf{A}$ are all real
   - $\mathbf{A}$ is diagonable
   - the eigenvectors are orthogonal to each other
   - the rank of $\mathbf{A}$ equals the number of nonzero eigenvalues
   - positive definite matrices have eigenvalues all greater than zero and vice versa

Diagonable Matrix

The matrix $\mathbf{A}$ is *diagonable* when a nonsingular matrix $\mathbf{X}$ exists, and consists of the *n* eigenvectors, that is,

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

where all *n* eigenvectors are linearly independent, such that

$$\mathbf{X}^{-1}\mathbf{AX} = \mathbf{D} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

Diagonable Matrix

For symmetric matrix $\mathbf{A}$, because the eigenvectors are orthogonal to each other, the above equation becomes

$$\mathbf{X'AX} = \mathbf{D} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

It follows that $\mathbf{A}$ can be written as $\mathbf{XDX'}$, or

$$\mathbf{A} = \lambda_1\mathbf{x}_1\mathbf{x}_1' + \lambda_2\mathbf{x}_2\mathbf{x}_2' + \ldots + \lambda_n\mathbf{x}_n\mathbf{x}_n'$$

Spectral Decomposition

The *spectral decomposition* of the matrix $\mathbf{A}$ is given by:

$$\mathbf{A} = \lambda_1\mathbf{x}_1\mathbf{x}_1' + \lambda_2\mathbf{x}_2\mathbf{x}_2' + \ldots + \lambda_n\mathbf{x}_n\mathbf{x}_n'$$

When $\mathbf{A}$ is nonsingular, the spectral decomposition of $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \lambda_1^{-1}\mathbf{x}_1\mathbf{x}_1' + \lambda_2^{-1}\mathbf{x}_2\mathbf{x}_2' + \cdots + \lambda_n^{-1}\mathbf{x}_n\mathbf{x}_n'$$