Repeated Measures Designs

Univariate Paired Comparisons

Subjects	Before	After	Average
1	y_{11}	y_{12}	$\bar{y}_{1\bullet}$
2	y_{21}	y_{22}	$\bar{y}_{2\bullet}$
⋮	⋮	⋮	⋮
n	y_{n1}	y_{n2}	$\bar{y}_{n\bullet}$
Average	$\bar{y}_{\bullet 1}$	$\bar{y}_{\bullet 2}$	$\bar{y}_{\bullet\bullet}$

Each subject is measured twice.
Therefore, this is actually a multivariate design.

The Multivariate General Linear Model for Univariate Paired Comparisons

$$\mathbf{Y} = \mathbf{X} \quad \mathbf{B} + \mathbf{E}$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ \vdots & \vdots \\ y_{N1} & y_{N2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \\ \vdots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} \end{bmatrix}$$

Paired Comparisons – Hypothesis

$$\mathrm{H}_0 : \mu_1 - \mu_2 = 0$$
$$\mathrm{H}_1 : \mu_1 - \mu_2 \neq 0$$

Or, framed in terms of the
Multivariate General Linear Hypothesis:

$$\mathrm{H}_0 : \mathbf{C} \quad \mathbf{B} \quad \mathbf{A} = \mathbf{\Gamma}$$

$$\mathrm{H}_0 : 1 \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Paired Comparisons
Test Statistic

Let $d_i = y_{i1} - y_{i2}$

Then $\mathrm{H} = N\bar{d}^{\,2}$

$$\mathrm{E} = (N-1)s_d^2$$

$$\mathrm{HE}^{-1} = \frac{t^2}{N-1}$$

$$t^2 \sim F_{1,\,N-1}$$

Multivariate Paired Comparisons

Subjects	Before	After	Average
1	$\mathbf{y}_{11}$	$\mathbf{y}_{12}$	$\bar{\mathbf{y}}_{1\bullet}$
2	$\mathbf{y}_{21}$	$\mathbf{y}_{22}$	$\bar{\mathbf{y}}_{2\bullet}$
⋮	⋮	⋮	⋮
n	$\mathbf{y}_{n1}$	$\mathbf{y}_{n2}$	$\bar{\mathbf{y}}_{n\bullet}$
Average	$\bar{\mathbf{y}}_{\bullet 1}$	$\bar{\mathbf{y}}_{\bullet 2}$	$\bar{\mathbf{y}}_{\bullet\bullet}$

Each subject is measured twice on a set of p variables. Therefore, this is a doubly multivariate design.

The Multivariate General Linear Model for Multivariate Paired Comparisons

$$\mathbf{Y} = \mathbf{X} \quad \mathbf{B} \quad + \mathbf{E}$$

$$\begin{bmatrix} y_{111} & \cdots & y_{11p} & y_{121} & \cdots & y_{12p} \\ y_{211} & \cdots & y_{21p} & y_{221} & \cdots & y_{22p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{N11} & \cdots & y_{N1p} & y_{N21} & \cdots & y_{N2p} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_{11} & \cdots & \mu_{1p} & \mu_{21} & \cdots & \mu_{2p} \end{bmatrix} + \mathbf{E}$$

The Multivariate General Linear Model for Multivariate Paired Comparisons

$$\mathbf{Y} = \mathbf{X} \quad \mathbf{B} \quad + \mathbf{E}$$

$$\begin{bmatrix} \mathbf{y}'_{11} & \mathbf{y}'_{12} \\ \mathbf{y}'_{21} & \mathbf{y}'_{22} \\ \vdots & \vdots \\ \mathbf{y}'_{N1} & \mathbf{y}'_{N2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}'_1 & \boldsymbol{\mu}'_2 \end{bmatrix} + \mathbf{E}$$

Multivariate Paired Comparisons Hypothesis

$$H_0: \begin{bmatrix} \mu_{11} - \mu_{21} \\ \mu_{12} - \mu_{22} \\ \vdots \\ \mu_{1p} - \mu_{2p} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$H_1: \begin{bmatrix} \mu_{11} - \mu_{21} \\ \mu_{12} - \mu_{22} \\ \vdots \\ \mu_{1p} - \mu_{2p} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multivariate Paired Comparisons General Linear Hypothesis

$$H_0: \mathbf{C} \quad \mathbf{B} \quad \mathbf{A} = \mathbf{\Gamma}$$

$$H_0: 1\begin{bmatrix} \boldsymbol{\mu}_1' & \boldsymbol{\mu}_2' \end{bmatrix} \begin{bmatrix} \mathbf{I}_p \\ -\mathbf{I}_p \end{bmatrix} = \mathbf{0}$$

Multivariate Paired Comparisons Test Statistic

Let $\mathbf{d}_i = \mathbf{y}_{i1} - \mathbf{y}_{i2}$

Then $\mathbf{H} = N\,\bar{\mathbf{d}}\,\bar{\mathbf{d}}'$

$\mathbf{E} = (N-1)\mathbf{S}_{\mathbf{d}}$

Multivariate Paired Comparisons Test Statistic

The multivariate test statistics are functions of $\mathbf{HE}^{-1}$

$$rank\left(\mathbf{HE}^{-1}\right) = 1$$

$$\lambda_1\left(\mathbf{HE}^{-1}\right) = \frac{\mathrm{T}_{\mathbf{d}}^2}{N-1}$$

$$\frac{(N-p)}{p(N-1)}\mathrm{T}_{\mathbf{d}}^2 \sim F_{p,\,N-p}$$

Simultaneous Confidence Intervals

A 100(1-α)% C.I. for $\mu_{1j} - \mu_{2j}$ is:

$$\bar{d}_j \pm c_0 \sqrt{\frac{s^2_{d_j}}{N}}$$

where c_0 is given by:

$$Hotelling's\ \mathrm{T}^2 : c_0^2 = \mathrm{T}^2_{p,N-1;\alpha} = \frac{p(N-1)}{N-p} F_{p,N-p;\alpha}$$

or $\quad Bonferroni : c_0 = t_{N-1;\alpha/2r}$

Univariate Repeated Measures

<u>One Within-Subjects Factor</u>

	Treatments				
Subjects	1	2		q	Average
1	y_{11}	y_{12}	⋯	y_{1q}	$\bar{y}_{1\bullet}$
2	y_{21}	y_{22}	⋯	y_{2q}	$\bar{y}_{2\bullet}$
⋮	⋮	⋮		⋮	⋮
n	y_{n1}	y_{n2}	⋯	y_{nq}	$\bar{y}_{n\bullet}$
Average	$\bar{y}_{\bullet 1}$	$\bar{y}_{\bullet 2}$	⋯	$\bar{y}_{\bullet q}$	$\bar{y}_{\bullet\bullet}$

One Within-Subjects Factor Univariate Analysis

Strictly Additive Model:

$$y_{ij} = \mu + \eta_i + \tau_j + \varepsilon_{ij} \qquad i = 1, \cdots, n;\ j = 1, \cdots, q$$

Hypotheses to be tested:

$$\mathrm{H}_0 : \mu_1 = \mu_2 = \cdots = \mu_q$$

or equivalently

$$\mathrm{H}_0 : \tau_1 = \tau_2 = \cdots \tau_q = 0$$

ANOVA Table showing Expected Mean Squares

Source	df	SS	MS	E(MS)
Between Subjects	n-1	SS(BS)	MS(BS)	$\sigma^2 + q\sigma_\eta^2$
Within Subjects	n(q-1)			
Treatments	q-1	SST	MST	$\sigma^2 + \sigma_{\eta\times\tau}^2 + n\sigma_\tau^2$
Residual	(n-1)(q-1)	SSR	MSR	$\sigma^2 + \sigma_{\eta\times\tau}^2$
Total	nq-1			

ANOVA Table

Source	df	SS	MS	F
Between Subjects	n-1	SS(BS)	MS(BS)	
Within Subjects	n(q-1)			
Treatments	q-1	SST	MST	$F = \frac{MST}{MSR}$
Residual	(n-1)(q-1)	SSR	MSR	
Total	nq-1	SS(Total)		

Constraints on the Covariance Matrix

The variance-covariance matrix for a design with one within-subjects factor with q repeated measures is:

$$\mathbf{\Sigma}_{q\times q} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1q} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{q1} & \sigma_{q2} & \cdots & \sigma_{qq} \end{bmatrix}$$

Constraints on the Covariance Matrix

In order for the F statistic to have an F distribution, this variance-covariance matrix must conform to certain patterns:

- Circularity (necessary & sufficient)
- Sphericity (sufficient)
- Compound Symmetry (sufficient)

Circularity of Σ

Circularity is defined by:

$$\sigma_{jj} + \sigma_{j'j'} - 2\sigma_{jj'} = 2\lambda \qquad \text{for all } j \neq j'$$

or equivalently

$$\sigma^2_{y_j - y_{j'}} = 2\lambda \qquad \text{for all } j \neq j'$$

or equivalently

$$\bar{\sigma}_{jj} - \bar{\sigma}_{jj'} = \lambda$$

Sphericity

Let $\mathbf{M}$ be an orthonormal matrix, i.e.,

$$\mathbf{M}\mathbf{M}' = \mathbf{I}$$

If $\mathbf{\Sigma}$ is circular, then

$$\mathbf{M}\mathbf{\Sigma}\mathbf{M}' = \lambda\mathbf{I}$$

A matrix having the form $\lambda\mathbf{I}$ is said to be spherical.

Compound Symmetry

A circular matrix $\mathbf{\Sigma}$ that also has the property that the variances are all equal to one another, and the covariances are all equal to one another, defines a matrix which has compound symmetry.

Compound Symmetry

$$
\begin{aligned}
\boldsymbol{\Sigma}_{q\times q} &= \sigma^2 \mathbf{P} \\
&= \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}
\end{aligned}
$$

Compound Symmetry

Compound symmetry may be a reasonable model when the repeated measures factor represents different levels of a treatment factor, and each subject receives the different treatments in random order.

Distribution of F

For an arbitrary variance-covariance matrix Σ, the distribution of the F statistic can be shown to be:

$$F \sim F_{\varepsilon(q-1),\, \varepsilon(q-1)(n-1)}$$

where ε can be viewed as a measure of departure from circularity, with

$$\frac{1}{q-1} \le \varepsilon \le 1$$

Distribution of F

When Σ is circular, $\varepsilon = 1$. As Σ departs from circularity, ε decreases, but never goes below $1/(q\text{-}1)$.

Methods of estimating ε have been proposed by

- Greenhouse-Geisser
- Huynh-Feldt

These corrections to the degrees of freedom are often available in software packages, and in particular, are available in SAS and SPSS.

Testing for Circularity

Testing for the circularity of Σ can be accomplished by applying a sphericity test to any set of variables defined by an orthogonal contrast transformation. Such a set of variables is known as a set of orthogonal components.

This test is available in SAS PROC GLM with the PrintE option in the REPEATED statement.

Additional Assumptions

Additional necessary assumptions for the F test to have an F distribution:

$$\varepsilon_{ij} \sim iid\ N\left(0, \sigma^2\right)$$

$$\eta_i \sim iid\ N\left(0, \sigma_\eta^2\right)$$

Multivariate Analysis of a Univariate Repeated Measures Design

- An alternative to the univariate analysis of a univariate repeated measures design with one within-subjects factor is a multivariate analysis.
- The multivariate analysis makes no assumption about the structure of the variance-covariance matrix Σ.

The Multivariate General Linear Model

$$\mathbf{Y} = \mathbf{X}\mathbf{B} + \mathbf{E}$$

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1q} \\ y_{21} & y_{22} & \cdots & y_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nq} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_q \end{bmatrix} + \mathbf{E}$$

Hypothesis

$$\mathrm{H}_0 : \mu_1 = \mu_2 = \cdots = \mu_q$$

or equivalently

$$\mathrm{H}_0 : \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_1 \\ \vdots \\ \mu_q - \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hypothesis

$$\mathrm{H}_0 : \begin{bmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_1 \\ \vdots \\ \mu_q - \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This formulation of the null hypothesis would be reasonable if the first measurement represented a control condition, and the other measurements arose from one of the other q-1 treatments.

General Linear Hypothesis

$$\mathrm{H}_0 : \mathbf{C} \quad \mathbf{B} \quad \mathbf{A} = \mathbf{\Gamma}$$

$$\mathrm{H}_0 : 1\begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_q \end{bmatrix}\begin{bmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{0}$$

Transformed Variables

Denote the (q-1) transformed variables as:

$$\mathbf{y}_d = \begin{bmatrix} y_2 - y_1 \\ y_3 - y_1 \\ \vdots \\ y_q - y_1 \end{bmatrix}$$

Transformed Variables

Let $\bar{\mathbf{y}}_d$ represent the sample mean vector, and $\mathbf{S}_d$ represent the sample covariance matrix of y_2-y_1, y_3-y_1, …, y_q-y_1.
Then

$$\mathbf{H} = n\bar{\mathbf{y}}_d\bar{\mathbf{y}}'_d$$
$$\mathbf{E} = (n-1)\mathbf{S}_d$$

Multivariate Test Statistics

Hotelling's T^2 is given by:

$$\mathrm{T}^2 = n\bar{\mathbf{y}}'_d\,\mathbf{S}_d^{-1}\,\bar{\mathbf{y}}_d \sim \mathrm{T}^2_{q-1,\,n-1}$$

The four multivariate test criteria are all equivalent to T^2, and all of these five statistics transform to an F statistic.

$$\mathrm{F} = \frac{n-q+1}{(n-1)(q-1)}\mathrm{T}^2 \sim \mathrm{F}_{q-1,\,n-q+1}$$

Nonuniqueness of the $\mathbf{A}$ matrix

The choice of the $\mathbf{A}$ matrix to test

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_q$$

is not unique. Another valid $\mathbf{A}$ matrix would be:

Nonuniqueness of the $\mathbf{A}$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

Simultaneous Confidence Intervals

A 100(1-α)% C.I. for $\mathbf{b}'\boldsymbol{\mu}$ is:

$$\mathbf{b}'\bar{\mathbf{y}} \pm c_0 \sqrt{\frac{\mathbf{b}'\mathbf{S}\mathbf{b}}{n}}$$

where c_0 is given by:

$$Hotelling's\ \mathrm{T}^2 : c_0^2 = \mathrm{T}^2_{q-1,n-1;\alpha} = \frac{(n-1)(q-1)}{n-q+1} F_{q-1,n-q+1;\alpha}$$

or $\quad Bonferroni : c_0 = t_{n-1;\alpha/2m}$

Repeated Measures
1 Within-Subjects Factor & 1 Between-Subjects Factor

Factor A	Subjects	Treatments 1	2	...	q
1	1	y_{111}	y_{112}	...	y_{11q}
	⋮	⋮	⋮		⋮
	n	y_{1n1}	y_{1n2}	...	y_{1nq}
2	1	y_{211}	y_{212}	...	y_{21q}
	⋮	⋮	⋮		⋮
	n	y_{2n1}	y_{2n2}	...	y_{2nq}
⋮	⋮	⋮	⋮		⋮
a	1	y_{1a1}	y_{a12}	...	y_{a1q}
	⋮	⋮	⋮		⋮
	n	y_{an1}	y_{an2}	...	y_{anq}

Repeated Measures – 1B & 1W Table of Means

	Treatments T				
Factor A	1	2	···	q	Average
1	μ_{11}	μ_{12}	···	μ_{1q}	$\mu_{1\bullet}$
2	μ_{21}	μ_{22}	···	μ_{2q}	$\mu_{2\bullet}$
⋮	⋮	⋮		⋮	⋮
a	μ_{a1}	μ_{a2}	···	μ_{aq}	$\mu_{a\bullet}$
Average	$\mu_{\bullet 1}$	$\mu_{\bullet 2}$	···	$\mu_{\bullet q}$	$\mu_{\bullet\bullet}$

Repeated Measures – 1B & 1W Model

$$y_{ijk} = \mu + \alpha_i + \eta_{j(i)} + \tau_k + (\alpha\tau)_{ik} + (\eta\tau)_{jk(i)} + \varepsilon_{ijk}$$

$$i = 1, \cdots, a$$

$$j = 1, \cdots, n$$

$$k = 1, \cdots, q$$

Hypothesis
Between-Subjects Factor

$$H_0: \mu_{1\bullet} = \mu_{2\bullet} = \cdots = \mu_{a\bullet}$$

or equivalently

$$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$$

Hypothesis
Within-Subjects Factor

$$H_0: \mu_{\bullet 1} = \mu_{\bullet 2} = \cdots = \mu_{\bullet q}$$

or equivalently

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_q = 0$$

Hypothesis
Interaction between Between-Subjects Factor and Within-Subjects Factor

$$H_0 : (\alpha\tau)_{ik} = 0 \qquad \forall i \neq k$$

ANOVA Table showing Expected Mean Squares

Source	df	SS	MS	E(MS)
Between Subjects				
A	a-1	SS[A]	MS[A]	$\sigma^2 + q\sigma_\eta^2 + nq\sigma_\alpha^2$
S(A)	a(n-1)	SS[S(A)]	MS[S(A)]	$\sigma^2 + q\sigma_\eta^2$
Within Subjects				
T	q-1	SS[T]	MS[T]	$\sigma^2 + q\sigma_{\eta\tau}^2 + an\sigma_\tau^2$
A×T	(a-1)(q-1)	SS[A×T]	MS[A×T]	$\sigma^2 + q\sigma_{\eta\tau}^2 + n\sigma_{\alpha\tau}^2$
T×S(A)	a(n-1)(q-1)	SS[T×S(A)]	MS[T×S(A)]	$\sigma^2 + q\sigma_{\eta\tau}^2$

ANOVA Table

Source	df	SS	MS	F
Between Subjects				
A	a-1	SS[A]	MS[A]	$F_A = \frac{MS_A}{MS_{S(A)}}$
S(A)	a(n-1)	SS[S(A)]	MS[S(A)]	
Within Subjects				
T	q-1	SS[T]	MS[T]	$F_T = \frac{MS_T}{MS_{T\times S(A)}}$
A×T	(a-1)(q-1)	SS[A×T]	MS[A×T]	$F_{A\times T} = \frac{MS_{A\times T}}{MS_{T\times S(A)}}$
T×S(A)	a(n-1)(q-1)	SS[T×S(A)]	MS[T×S(A)]	

Assumptions for the F test to be valid

$$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \cdots = \mathbf{\Sigma}_a = \mathbf{\Sigma}$$

$\mathbf{\Sigma}$ circular

$$\varepsilon_{ijk} \sim iid\ N\left(0, \sigma^2\right)$$

$$\eta_{j(i)} \sim iid\ N\left(0, \sigma_\eta^2\right)$$

Repeated Measures
2 Within-Subjects Factors

	T									
	1			2			...	q		
	U			U				U		
Subjects	1	...	r	1	...	r		1	...	r
1	y_{111}	...	y_{11r}	y_{121}	...	y_{12r}	...	y_{1q1}	...	y_{1qr}
:	:		:	:		:		:		:
n	y_{n11}	...	y_{n1r}	y_{n21}	...	y_{n2r}	...	y_{nq1}	...	y_{nqr}

ANOVA Table showing
Expected Mean Squares

Source	df	SS	MS	E(MS)
Within Subjects	n(q-1)			
T	q-1	SS[T]	MS[T]	$\sigma^2 + r\sigma^2_{\eta\times\tau} + nr\sigma^2_{\tau}$
T×S	(n-1)(q-1)	SS[T×S]	MS[T×S]	$\sigma^2 + r\sigma^2_{\eta\times\tau}$
U	r-1	SS[U]	MS[U]	$\sigma^2 + q\sigma^2_{\eta\times\upsilon} + nq\sigma^2_{\upsilon}$
U×S	(n-1)(r-1)	SS[U×S]	MS[U×S]	$\sigma^2 + \sigma^2_{\eta\times\upsilon}$
T×U	(q-1)(r-1)	SS[T×U]	MS[T×U]	$\sigma^2 + \sigma^2_{\eta\times\tau\times\upsilon} + n\sigma^2_{\tau\times\upsilon}$
T×U ×S	(n-1)(q-1)(r-1)	SS[T×U ×S]	MS[T×U ×S]	$\sigma^2 + \sigma^2_{\eta\times\tau\times\upsilon}$
Total	nqr-1	SS(Total)		

ANOVA Table

Source	df	SS	MS	F
Within Subjects	n(q-1)			
T	q-1	SS[T]	MS[T]	$F_T = \frac{MS_T}{MS_{T\times S}}$
T×S	(n-1)(q-1)	SS[T×S]	MS[T×S]	
U	r-1	SS[U]	MS[U]	$F_U = \frac{MS_U}{MS_{U\times S}}$
U×S	(n-1)(r-1)	SS[U×S]	MS[U×S]	
T×U	(q-1)(r-1)	SS[T×U]	MS[T×U]	$F_{T\times U} = \frac{MS_{T\times U}}{MS_{T\times U\times S}}$
T×U ×S	(n-1)(q-1)(r-1)	SS[T×U ×S]	MS[T×U ×S]	
Total	nqr-1	SS(Total)		

Assumptions for the F test to be valid

- The $q\times q$ covariance matrix associated with the q levels of Factor T is circular.
- The $r\times r$ covariance matrix associated with the r levels of Factor U is circular.
- The $qr\times qr$ covariance matrix associated with the qr levels of Factor T and U together is circular.
- Independence of observations
- Normality of observations.