The General Linear Hypothesis

With the General Linear Model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Suppose we have 3 predictor variables x_1, x_2, x_3, so p=4.

Then
$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The ANOVA table for this model would produce an omnibus test of the hypothesis

$$H_0: \beta_1 = \beta_2 = \beta_3 = 0$$

But we would typically be interested in testing other hypotheses involving a subset of the β's, e.g.,

$$H_0: \beta_1 = 0$$

or

$$H_0: \beta_2 = 0$$

or

$$H_0: \beta_3 = 0$$

or

$$H_0: \beta_1 = \beta_2 = 0$$

or

$$H_0: \beta_1 = \beta_2$$

The hypothesis

$$H_0: \beta_1 = 0$$

is equivalent to

$$H_0 : \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = 0$$

or

$$H_0 : \mathbf{C\beta = 0}$$

The hypothesis

$$H_0: \beta_1 = \beta_2 = 0$$

is equivalent to

$$H_0: \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$H_0: \mathbf{C\beta} = \mathbf{0}$$

The hypothesis

$$H_0: \beta_1 = \beta_2$$

or

$$H_0: \beta_1 - \beta_2 = 0$$

is equivalent to

$$H_0 : \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = 0$$

or

$$H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{0}$$

Suppose we want to test

$$H_0 : \mathbf{C\beta = 0}$$
$$H_1 : \ \sim H_0$$

Assume the model

$$E(\mathbf{y}) = \mathbf{X\beta}$$

is correct

So,

$$\mathbf{b = (X'X)^{-1} X'y} \sim N\left(\boldsymbol{\beta}, \left(\mathbf{X'X}\right)^{-1} \sigma^2\right)$$

$$SS_E = \mathbf{y'y - b'X'y}$$

has $n - p$ degrees of freedom

Assume $\mathbf{C}$ is $q \times p$ with rank q

Then the test statistic for testing the General Linear Hypothesis is

$$F = \frac{(\mathbf{Cb})^{\mathbf{-1}}\left(\mathbf{C}(\mathbf{X}'\mathbf{X})^{\mathbf{-1}}\mathbf{C}'\right)^{\mathbf{-1}}(\mathbf{Cb})/q}{s^2}$$

$$\sim F_{q,n-p}$$

Example:

Assume $p = 4$,

$$E(\mathbf{y}) = \mathbf{X\beta}$$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i \quad i = 1, \ldots, n$$

Want to test

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

$$H_1 : \ \sim H_0$$

This can be expressed as

$$H_0 : \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H_0 : \begin{bmatrix} \beta_1 - \beta_2 \\ \beta_2 - \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example:

Still assume $p = 4$

Want to test

$$H_0 : \beta_1 = \beta_2 = \beta_3 = 0$$

$$H_1 : \ \sim H_0$$

This can be expressed as

$$H_0 : \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H_0 : \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$