Resolution IV Designs with 128 Runs

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Abstract

Chen, Sun and Wu (1993) enumerated all possible 2^{k-p} fractional factorial designs of size 16 and 32, and all resolution four (or higher) fractions of size 64. By enumerating all possible designs, they not only provided the minimum aberration design for each value of k, but also listed designs attractive for other reasons, e.g., having the most clear two-factor interactions. Here we present the results of an enumeration of n = 128 run resolution IV designs. As in Chen, Sun and Wu (1993), we constructed new designs by building up, adding one factor at a time. However, rather than determining whether a new candidate design was isomorphic to an existing design based on a complete permutation check, we retained all designs that differed in their projections. Resolution IV designs are tabulated for k = 12, …, 64 factors.

Key words and phrases: Alias length pattern, Hamming distance, isomorphism, projection, row coincidence matrix, weak minimum aberration, word length pattern.

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Introduction

Regular resolution IV fractional factorial designs of size 128 exist for up to 64 factors. For $k \le 11$ factors, regular fractional factorial designs of resolution V or more exist. Hence, this article focuses on enumerating the best 128-run, 2_{IV}^{k-p} designs for k = 12, …, 64. (For nonregular resolution V designs with 12-15 factors, see the Conclusion.) Chen, Sun, and Wu (1993) enumerated all regular 2_{IV}^{k-p} designs of size n = 64 and smaller, and provided tables of the best designs. Recently, Butler (2003) provided some theory that simplifies the search for the best even 2_{IV}^{k-p} designs. A resolution IV design is even if all the words in its defining contrast subgroup are of even length. For $k > 5n/16$, all resolution IV designs are even. For fewer factors, better resolution IV designs will contain both even-length and odd-length words in the defining contrast subgroup; we refer to such designs as even/odd. The next section will provide a summary of the best 2_{IV}^{k-p} designs (all of which are even/odd) for k = 12, …, 40. The subsequent section contains an enumeration of the best 2_{IV}^{k-p} designs for k = 41, …, 64. The final section summarizes our work and its potential applications. Technical details are provided in the Appendix.

Several criteria have been proposed for comparing resolution IV designs. The criteria we find the most useful are based on the word length pattern and alias length pattern (Block and Mee 2003). The minimum aberration criterion of Fries and Hunter (1980) is based on the word length pattern. For resolution IV designs, the word length pattern is the vector wlp = (w_4, w_5, …, w_k), where w_i is the number of words of length i in the defining contrast subgroup. For cases where resolution IV designs have maximal resolution, a 2_{IV}^{k-p} design has weak minimum aberration (Chen and Hedayat 1996) if no other resolution IV design has fewer length-four words. If there exist several designs that minimize w_4, the minimum aberration design is determined by sequentially comparing designs with respect to w_5, w_6, etc., until the tie is broken. We have discovered instances where the mimimum aberration design is not unique; that is, there exist two or more mimimum aberration designs with identical wlp.

Chen, Sun and Wu (1993) call attention to the fact that one might prefer a design that is not minimum aberration if, e.g., it has more clear two-factor interactions that the minimum aberration design. We capture the number of clear two-factor interactions (2fi's) and other useful information through the alias length pattern, defined as alp = (a_1, a_2, … , a_L), where a_1 is the number of two-factor interactions clear of aliasing with other 2fi's, a_2 = number of pairs of aliased 2fi's, …, and a_L = number of chains of aliased 2fi's of length L, where L is the length of the longest chain of aliased 2fi's.

In addition to ranking designs based on wlp, a_1, and L, we compute the number of degrees of freedom allocated to main effects and two-factor interactions, namely, df = k + $a_1 + \cdots + a_L$. Whenever df = n-1, we refer to the design as a second-order saturated (sos) design (Block and Mee 2003). There is only one even sos design for each n, the unique resolution IV design with $k = n/2$. All other even resolution IV designs of size n are projections of this design. There are 87 even/odd resolution IV sos designs at n = 128 (see Block 2003). All other even/odd resolution IV designs of size 128 are projections of one or more of these.

We now illustrate these ideas with one of the four sos designs at *n* = 128 that is minimum aberration – the 2_{IV}^{29-22} design 29-22.1 from Tables 1 and 2, with w_4= 266 (21 fewer than the next best design), L = 7 (better than any other design), and alp = (0,0,70,0,0,0,28). Note that this design does not have any clear two-factors but in other respects is favored above all 29-factor designs of size 128. In addition, it is possible to delete one, two, three, or five factors from this design and obtain the minimum aberration design for *k* = 28, 27, 26, or 24 factors, respectively. By arranging the last few generators for this sos design in a particular order, one can thus provide five of the minimum aberration designs in a compact fashion. In a similar manner, the minimum aberration designs for *k* = 30, 31, …, 39 may be obtained by sequentially eliminating factors from the 40-33.1 design. Details are provided in the next section.

Even/Odd Resolution IV Designs

Block and Mee (2003) list all even/odd resolution IV designs of size 64 and show their even/odd projections. As we produced this summary, we recognized that every 2_{IV}^{k-p} design of size 64 has a unique set of delete-one-factor projections. For instance, Block and Mee's design 9-3.b with generators G=ABC, H=ADE, and J=ABDF (and defining contrast subgroup ABCG, ADEH, BCDEGH, ABDFJ, CDFGJ, BEFHJ, ACEFGHJ) has the following nine delete-one-factor projections:

- Design 8-2.a with wlp=(0,2,1) if one deletes factor A
- Design 8-2.b with wlp=(1,1,0,1) if one deletes factor B or D
- Design 8-2.c with wlp=(1,2) if one deletes factor C, E, G, or H
- The even design with wlp=(2,0,1) if one deletes factor F or J

For $n = 128$, we used the set of delete-one-factor projections to discriminate between resolution IV designs. That is, we declare two designs to be isomorphic if their sets of delete-one-factor projections are equivalent. (Note that if the delete-one-factor projections are equivalent, then all subsequent projections are also equivalent.) Our enumeration of resolution IV designs at $n = 128$ assumes that if two designs have equivalent delete one-factor projection sets, then there exists a permutation of the columns and rows of one design to make it identical to the other. If this conjecture is false at $n = 128$, then we have not enumerated all possible even/odd designs. However, the set of delete-one-factor projections was more discriminating that alp, wlp, letter pattern, or any other function that we implemented (for more details, refer to the Appendix).

Table 1 shows the abbreviated wlp (w_4, w_5, w_6 only), the alp, and df for each of the minimum aberration and weak minimum aberration designs obtained in this manner for $k = 12, \ldots 40$. The generators (identified by Yates order) are given for these designs in Table 2. For more complete tables showing the best designs according to wlp, a_1, L, or df, see Block (2003). Regarding Yates order used to denote generators in Table 2 (and subsequently in Tables 4 and 5), columns 1, 2, 4, 8, 16, 32, and 64 are the *basic columns* (denoted X1, …, X7 below) from which all generators for factors X8, X9, … are defined. Every generator is a product of three or more of the seven basic columns. Block (2003, pp. 75-76) shows the Yates order for columns 1-127. Rather than reproduce that table here, recognize that the Yates column number for any generator identifies the basic columns involved, since the column number is the sum of those basic column numbers. For instance, since 7 = 1 + 2 + 4, the generator corresponding to column 7 is the product of the first three basic columns, X1-X3. Further examples are given in the next paragraph.

As mentioned in the previous section, ten minimum aberration designs may be obtained by sequentially deleting factors from the minimum aberration design at *k* = 40. In particular, the minimum aberration 2_{IV}^{30-23} design is obtained using (Yates order) columns 23, 25, 26, 28, 39, 43, 45, 46, 51, 53, 56, 63, 71, 73, 74, 76, 81, 84, 88, 99, 101, 102, 104, 112 as generators for factors, X8, …, X30, respectively. The minimum aberration designs for *k* = 31, …, 40 factors are obtained by sequentially adding:

- X31 = X3*X4*X5 (column 28, since 28 = 4 + 8 + 16)
- X32 = X2*X5*X7 (column 82, since 82 = 2 + 16 + 64)
- X33 = X2*X3*X5*X6 (column 54, since 54 = 2 + 4 + 16 + 32)
- X34 = X1*X2*X3*X4*X5*X7 (column 95, since 95 = 1 + 2 + 4 + 8 + 16 + 64)
- X35 = X1*X2*X3*X4*X6*X7 (column 111, since 95 = 1 + 2 + 4 + 8 + 32 + 64)
- X36 = X1*X2*X3*X4 (column 15, since 15 = 1 + 2 + 4 + 8)
- X37 = X1*X2*X3*X5*X6*X7 (column 119, since 119 = 1 + 2 + 4 + 16 + 32 + 64)
- X38 = X1*X2*X4*X5*X6*X7 (column 123, since 119 = 1 + 2 + 8 + 16 + 32 + 64)
- X39 = X1*X3*X4*X5*X6*X7 (column 125, since 119 = 1 + 4 + 8 + 16 + 32 + 64)
- X40 = X2*X3*X4*X5*X6*X7 (column 126, since 119 = 2 + 4 + 8 + 16 + 32 + 64)

To summarize the results for even/odd resolution IV designs, the minimum aberration designs at k = 40, 29, and 25 are sos designs. Naively projecting these produces the remaining minimum aberration designs for k = 24-39:

- As just shown, minimum aberration designs for *k* = 39, …, 30 are obtained by sequentially deleting factors from the sos design at *k* = 40
- As mentioned in the introduction, minimum aberration designs for *k* = 28, 27, 26, 24 may be obtained by sequentially deleting factors from the sos design at *k* = 29. In

particular, deleting sequentially deleting columns 69, 60, 97, and the pair (70, 94), respectively produces the minimum aberration designs for k = 28, 27, 26, and 24.

- The minimum aberration design for k = 25 is sos. Minimum aberration designs for $k \leq 23$ are generally projections of sos designs that are not themselves minimum aberration. For details, see Block (2003, pp. 50-54).

Even Resolution IV Designs

For $k > 5n/16$, all resolution IV designs are projections of the resolution IV design at $k = n/2$. Butler (2003) showed that if the d columns omitted from the $n/2$ factor design are themselves a minimum aberration design, then the remaining $k = n/2 - d$ factor design has minimum aberration. Let $[D_k, D_d]$ denote any partitioning of the columns of the $n/2$-factor resolution IV design into subsets of size k and $d = n/2 - k$. Then, the difference in the number of length four words in the defining contrast subgroups for the two designs is

$$w_4 \text{ (for } k \text{ factor design)} - w_4 \text{ (for } d \text{ factor design)} = [2(k^4-d^4)/n - k(3k-2) + d(3d-2)]/24.$$

Since this difference is a constant, as Butler (2003) notes, one obtains the minimum aberration design for $k > 5n/16$ by selecting the $d < 3n/16$ factor even design with the minimum w_4.

We enumerated all even designs with the minimum w_4 for n = 128 and $d \leq 23$, discriminating designs based on their sets of delete-one-factor projections, in order to produce an enumeration of all (weak) minimum aberration resolution IV designs for $k \geq 41$. All designs that minimize the number of length-four words are listed in Table 3. Generators for these designs are given in Table 4. Note that, for k = 41, 42, 43, 44 and 50, the minimum aberration design is not unique. In some cases the designs differ on alp.

However, at k = 43, a pair of minimum aberration designs exist with identical alp and identical letter pattern matrices. The designs differ in their projections and in their row coincidence matrices (refer to the Appendix), but only in a subtle way.

Since all these designs are projections of a single design, it is helpful to arrange the generators in an order so that one may obtain a good k-factor design by simply taking the first k columns (i.e., the seven basic columns and the first k-7 generators). We have followed such an ordering in Table 4 for the 64-57.1 design. In only two cases for k = 63, 62, …, 44 is a minimum aberration design not obtained by deleting the last d = 64 – k generators. Those two exceptions are for k = 56, where the sequence includes a weak minimum aberration design, and k = 51. For k = 41, 42, and 43, the minimum aberration designs are not unique, and are not projections of design 44-37.1a. Thus, in Table 4, no attempt is made to identify the relations among these designs.

Discussion

We have presented tables of (weak) minimum aberration resolution IV designs of size 128 for 12-40 factors. For k = 12, …, 15, there actually exist 128-run nonregular orthogonal designs which permit estimation of all main effects and two-factor interactions with full efficiency (Hedayat, Sloane, and Stufken, 1999, p. 103). These nonregular designs can also be run in eight blocks of size 16 (for details, see Mee 2004). The regular designs enumerated in Tables 1 - 4 minimize the number of length four words in the defining contrast subgroup. For k > 15, these designs are generally the most attractive. However, since other criteria (e.g., minimizing the length of alias chains or maximizing the number of clear two-factor interactions) may result in a different ranking of designs, it is valuable to consider designs that are not minimum aberration. Table 5 provides a small list of designs that, while not weak minimum aberration, are preferred on other grounds. A fuller listing of alternative designs is available from Block (2003).

The designs listed in Table 5 are identified according to the '*k-p.i*' numbering in Block (2003), where the index *i* reflects ordering based on wlp. Although each design in Table 5 has one or two more length-four words than the corresponding minimum aberration design, they may be preferred on other grounds. Design 22-15.3 minimizes L; all the other designs in Table 5 have more clear two-factor interactions and larger df than the alternative designs in Table 1. In fact, designs 18-11.3, 21-14.8 and 24-17.3 have the maximum possible df.

For k = 15, …, 33, designs that maximize the number of clear two-factor interactions often have excessive w_4. For example, there exists a 2_{IV}^{21-14} with 84 clear two-factor interactions, but w_4 = 112 (Block 2003, p. 137, design 21-14.80683). The alternative designs in Tables 1 and 5 have w_4 = 51 and 52, respectively, but only 26-36 clear 2fi's. When we are interested in estimating a specific subset of two-factor interactions designs with large number of clear 2fi's need to be considered, since they may minimize the potential bias resulting from two-factor interactions omitted from the model. That is, in spite of their excessive w_4, designs with large C2fi will be minimum N-aberration designs (Ke and Tang 2003) for certain subsets of interactions.

The tables of Chen, Sun, and Wu (1993) for smaller fractional factorial designs have been of repeated use to practitioners and researchers. We hope that the resolution IV design tables presented here and in Block (2003) will be similarly useful. While we the minimum aberration designs listed here will be incorporated into statistical design software quickly, a full enumeration of designs will remain necessary to provide the best designs for more customized applications (for example, applications with run-order restrictions, or specified sets of effects to be estimated).

Appendix

Sun (2001) has provided all 92 possible n = 128 designs of resolution IV or more for k = 11. For each of these 92 designs, there are many available columns for an additional generator to produce a 2_{IV}^{12-5} fraction. Each of these possible designs was constructed, and its set of delete-one-factor projections identified. There were 249 such designs with distinct projection sets. For each $n \times k$ design matrix D, we used functions of the row-coincidence matrix $T = DD'$ (functions invariant to permutations of the rows of D) to compactly identify each non-isomorphic design. For more details, see Block (2003). In this manner, we sequentially enumerated all even/odd resolution IV designs for k = 12, …, 40, and all even resolution IV designs for $k \le 22$. (For k = 23, we calculated the minimum w_4 for all possible even designs, but completely discriminated only those designs with the minimum w_4 = 133. Table 6 provides a summary of the number of designs found for each k, separating even versus even/odd resolution IV designs. For the even/odd designs, we show the number of distinct word length patterns to indicate how inadequately wlp differentiates among the possible designs. The maximum number of even/odd designs occurs at k = 21 (although there are more distinct wlp at k = 22). For even designs, the maximum is anticipated at k = 32. Since at k = 22, the number of even designs is nearly doubling, we estimate that there are between five and 15 million non-isomorphic even designs for k = 12, …, 64.

References

BLOCK, R.M. (2003). *Theory and Construction Methods for Large Regular Resolution IV Designs.* University of Tennessee Ph.D. dissertation: Knoxville. [Available at http://etd.utk.edu/2003/BlockRobert.pdf]

BLOCK, R.M. and MEE, R.W. (2003). "Second Order Saturated Resolution IV Designs". *Journal of Statistical Theory and Applications* 2, pp.96-112.

BUTLER, N.A. (2003). "Some Theory for Constructing Minimum Aberration Fractional Factorial Designs". *Biometrika* 90, pp. 233-238.

CHEN, H. and HEDAYAT, A.S. (1996). "2^{n-m} Fractional Factorial Designs with Weak Minimum Aberration." *Annals of Statistics* 24, pp. 2536-2548.

CHEN, J., SUN, D.X., and WU, C.F.J. (1993). "A Catalogue of Two-Level and Three-Level Fractional Factorial Designs with Small Runs". *International Statistical Review* 61, 131-145.

FRIES, A. and HUNTER, W.G. (1980). "Minimum Aberration 2^{k-p} Designs". *Technometrics* 22, pp. 601-608.

HEDAYAT, A.S., SLOANE, N.J.A., and STUFKEN, J. (1999). *Orthogonal Arrays.* Springer: New York.

KE, W. and TANG, B. (2003). "Selecting 2^{m-p} Designs Using a Minimum Aberration Criterion When Some Two-Factor Interactions are Important". *Technometrics* 45, pp. 352-360.

MEE, R.W. (2004). "Efficient Two-Level Designs for Estimating All Main Effects and Two-Factor Interactions: A Review". *Journal of Quality Technology* 36, in press.

SUN, D.X. (2001). "Catalog of n = 64, 128 two-level designs". Private communication.

Table 1: Weak Minimum Aberration Regular Resolution IV Designs, n = 128, k = 12, …, 40

Design	abbreviated wlp w_4	w_5	w_6	alias length pattern	df	C2fi	L	df rank	C2fi rank	L rank
12-5.1	1	8	12	60 3	75	60	2	1	1	1
12-5.2	1	10	10	60 3	75	60	2			
12-5.3	1	10	11	60 3	75	60	2			
13-6.1	2	16	18	66 6	85	66	2	1	1	1
13-6.2	2	16	20	66 6	85	66	2			
14-7.1	3	24	36	73 9	96	73	2	1	1	1
15-8.1	7	32	52	63 21	99	63	2	2	11	1
15-8.2	7	34	46	63 21	99	63	2			
15-8.3	7	38	44	69 15 2	101	69	3			
16-9.1	10	48	72	60 30	106	60	2	2	24	1
17-10.1	15	60	130	46 45	108	46	2	53	1594	1
17-10.2	15	66	110	52 39 2	110	52	3			
17-10.3	15	68	106	52 39 2	110	52	3			
17-10.4	15	72	102	58 33 4	112	58	3			
18-11.1	20	80	200	33 60	111	33	2	209	10601	1
18-11.2	20	92	160	45 48 4	115	45	3			
19-12.1	27	120	235	36 54 9	118	36	3	22	5807	1
20-13.1	36	152	340	24 60 14 1	119	24	4	111	28084	1
21-14.1	51	200	414	26 54 15 4 3	123	26	5	23	17819	45
21-14.2	51	202	400	28 51 12 11	123	28	4			
22-15.1	65	248	572	25 36 32 8 0 1	124	25	6	20	14585	942
22-15.2	65	256	552	12 68 12 6 1 3	124	12	6			
23-16.1	83	316	744	12 52 24 9 2 2 1	125	12	7	10	32307	5495
23-16.2	83	318	734	14 54 11 17 6 1	125	14	5			

24-17.1	102	384	992	0 54 16 24 0 4	122	0	6	120	27865	4
24-17.2	102	394	985	7 57 9 17 12	126	7	5			
25-18.1	124	482	1312	0 64 0 18 20	127	0	5	1	20240	1
26-19.1	152	568	1704	0 29 41 4 16 8	124	0	6	13	13068	1
27-20.1	180	690	2200	0 15 55 0 12 16	125	0	6	6	7696	1
28-21.1	210	840	2800	0 0 70 0 0 28	126	0	6	2	3930	1
29-22.1	266	945	3472	0 0 70 0 0 0 28	127	0	7	1	1914	1
30-23.1	335	972	4662	0 0 0 40 40 0 0 0 0 2 5	117	0	11	773	799	182
31-24.1	391	1134	5826	0 0 0 24 48 8 0 0 0 0 3 4	118	0	12	323	331	96
31-24.2	391	1134	5827	0 0 0 24 48 8 0 0 0 0 3 4	118	0	12			
32-25.1	452	1322	7219	0 0 0 12 48 19 1 0 0 0 0 4 3	119	0	13	130	125	46
32-25.2	452	1323	7218	0 0 0 12 48 19 1 0 0 0 0 4 3	119	0	13			
32-25.3	452	1324	7219	0 0 0 12 48 19 1 0 0 0 0 4 3	119	0	13			
33-26.1	518	1543	8863	0 0 0 4 40 33 3 0 0 0 0 0 5 2	120	0	14	67	67	27
33-26.2	518	1544	8863	0 0 0 4 40 33 3 0 0 0 0 0 5 2	120	0	14			
34-27.1	589	1800	10788	0 0 0 0 24 50 6 0 0 0 0 0 0 6 1	121	-	15	11	-	1
34-27.2	589	1801	10788	0 0 0 0 24 50 6 0 0 0 0 0 0 6 1	121	-	15			
35-28.1	665	2100	13020	0 0 0 0 0 70 10 0 0 0 0 0 0 0 7	122	-	15	3	-	1
35-28.2	665	2101	13020	0 0 0 0 0 70 10 0 0 0 0 0 0 0 7	122	-	15			
36-29.1	756	2401	15736	0 0 0 0 0 42 38 0 0 0 0 0 0 0 7	123	-	16	2	-	1
37-30.1	854	2744	18886	0 0 0 0 0 21 51 8 0 0 0 0 0 0 0 7	124	-	17	1	-	1
38-31.1	959	3136	22512	0 0 0 0 0 7 49 24 0 0 0 0 0 0 0 0 7	125	-	18	1	-	1
39-32.1	1071	3584	26656	0 0 0 0 0 0 32 48 0 0 0 0 0 0 0 0 0 7	126	-	19	1	-	1
40-33.1	1190	4096	31360	0 0 0 0 0 0 0 80 0 0 0 0 0 0 0 0 0 0 7	127	-	20	1	-	1

wlp – word length pattern
alp – alias length pattern of two-factor interactions
df – total degrees of freedom used for main effects and two-factor interactions
C2fi – clear two-factor interactions
L – length of longest alp chain

Table 2: Generators for Table 1's (Weak) Minimum Aberration Resolution IV Designs, *n* = 128

Design	Generators
12-7.1	7, 57, 90, 108, 119
12-5.2	7, 27, 45, 78, 121
12-5.3	7, 27, 45, 86, 120
13-6.1	7, 27, 43, 85, 102, 120
13-6.2	7, 27, 43, 53, 78, 120
14-7.1	7, 27, 43, 53, 78, 118, 120
15-8.1	7, 27, 42, 53, 78, 83, 111, 120
15-8.2	7, 27, 42, 53, 75, 87, 116, 120
15-8.3	7, 11, 29, 45, 51, 78, 118, 120
16-9.1	7, 25, 42, 53, 75, 87, 108, 118, 120
17-10.1	7, 25, 42, 53, 62, 78, 83, 92, 99, 120
17-10.2	7, 11, 29, 45, 51, 78, 81, 100, 118, 120
17-10.3	Same as design 17-10.2 except replace 118 with 62
17-10.4	7, 11, 25, 45, 51, 62, 78, 84, 90, 120
18-11.1	Same as for design 17-10.1, plus 111
18-11.2	Same as for design 17-10.4, plus 101

19-12.1	7, 11, 21, 41, 54, 78, 79, 86, 92, 99, 101, 120
20-13.1	Same as for design 19-12.1, plus 123
21-14.1	7, 14, 25, 42, 54, 61, 69, 88, 104, 112, 121, 122, 124, 127
21-14.2	7, 30, 35, 38, 41, 52, 81, 82, 104, 112, 121, 122, 124, 127
22-15.1	7, 11, 19, 29, 37, 41, 55, 59, 74, 82, 84, 102, 108, 120, 126
22-15.2	7, 11, 19, 30, 38, 41, 52, 61, 74, 87, 93, 101, 111, 114, 120
23-16.1	7, 11, 19, 25, 26, 31, 35, 45, 46, 77, 81, 92, 100, 106, 118, 120
23-16.2	7, 11, 19, 30, 38, 57, 60, 70, 73, 76, 84, 93, 99, 110, 118, 120
24-17.1	7, 11, 19, 29, 35, 46, 53, 57, 73, 76, 82, 87, 100, 109, 118, 120, 123
24-17.2	Same as for design 23-16.2, plus 81
25-18.1	7, 11, 19, 29, 37, 41, 47, 49, 55, 59, 62, 77, 78, 82, 84, 91, 102, 120
26-19.1	7, 11, 19, 29, 35, 46, 53, 57, 70, 73, 76, 82, 87, 94, 100, 109, 118, 120, 123
27-20.1	Same as for design 26-19.1, plus 97
28-21.1	Same as for design 27-20.1, plus 60
29-22.1	Same as for design 28-21.1, plus 69
30-23.1	23, 25, 26, 28, 39, 43, 45, 46, 51, 53, 56, 63, 71, 73, 74, 76, 81, 84, 88, 99, 101, 102, 104, 112
31-24.1	Same as for design 30-23.1, plus 28

31-24.2	7, 11, 19, 29, 30, 35, 41, 44, 47, 53, 54, 56, 67, 77, 78, 81, 84, 88, 104, 112, 121, 122, 124, 127
32-25.1	Same as for design 31-24.1, plus 82
32-25.2	7, 11, 19, 29, 30, 35, 41, 42, 47, 53, 54, 56, 59, 77, 82, 84, 88, 102, 104, 107, 112, 121, 122, 124, 127
32-25.3	Same as design 32-25.2 except replace 29 with 101
33-26.1	Same as for design 32-25.1, plus 54
33-26.2	Same as for design 32-25.2, plus 44
34-27.1	Same as for design 33-26.1, plus 95
34-27.2	7, 11, 19, 29, 30, 35, 45, 46, 53, 54, 57, 58, 60, 63, 67, 77, 86, 89, 97, 98, 100, 103, 104, 107, 112, 115, 125
35-28.1	Same as for design 34-27.1, plus 111
35-28.2	15, 23, 25, 26, 28, 39, 43, 45, 46, 51, 53, 54, 56, 71, 73, 74, 76, 81, 82, 84, 88, 101, 104, 111, 112, 119, 123, 126
36-29.1	Same as for design 35-28.1, plus 15
37-30.1	Same as for design 36-29.1, plus 119
38-31.1	Same as for design 37-30.1, plus 123
39-32.1	Same as for design 38-31.1, plus 125
40-33.1	Same as for design 39-32.1, plus 126

Table 3: Weak Minimum Aberration Regular Resolution IV Designs, *n* = 128, k = 41, …, 64

Design	Abbreviated wlp		Alias Length Pattern	df	L
	w_4	w_6			
41-34.1a	1648	70146	0 0 0 0 0 0 0 0 0 0 0 14 37 9 3	104	15
41-34.1b	1648	70146	0 0 0 0 0 0 0 0 0 0 0 14 37 9 3	104	15
41-34.1c	1648	70146	0 0 0 0 0 0 0 0 0 0 0 16 31 15 1	104	15
41-34.1d	1648	70146	0 0 0 0 0 0 0 0 0 0 0 16 31 15 1	104	15
41-34.2a	1648	70146	0 0 0 0 0 0 0 0 0 0 0 14 37 9 3	104	15
41-34.2b	1648	70146	0 0 0 0 0 0 0 0 0 0 0 16 31 15 1	104	15
41-34.2c	1648	70146	0 0 0 0 0 0 0 0 0 0 0 16 31 15 1	104	15
41-34.2d	1648	70146	0 0 0 0 0 0 0 0 0 0 0 16 31 15 1	104	15
42-35.1a	1822	81828	0 0 0 0 0 0 0 0 0 0 0 0 27 30 6	105	15
42-35.1b	1822	81828	0 0 0 0 0 0 0 0 0 0 0 0 27 30 6	105	15
42-35.1c	1822	81828	0 0 0 0 0 0 0 0 0 0 0 0 27 30 6	105	15
43.36.1a	2009	95095	0 0 0 0 0 0 0 0 0 0 0 0 0 42 21	106	15
43-36.1b	2009	95095	0 0 0 0 0 0 0 0 0 0 0 0 0 42 21	106	15
44-37.1a	2214	110032	0 0 0 0 0 0 0 0 0 0 0 0 0 12 38 13	107	16
44-37.1b	2214	110032	0 0 0 0 0 0 0 0 0 0 0 0 0 12 38 13	107	16
44-37.1c	2214	110032	0 0 0 0 0 0 0 0 0 0 0 0 0 12 38 13	107	16
45-38.1	2430	126960	0 0 0 0 0 0 0 0 0 0 0 0 0 0 18 45	108	16
46-39.1	2665	145932	0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 30 30	109	17
47-40.1	2915	167244	0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 1 45 15	110	18
48-41.1	3180	191136	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 0 60	111	18
49-42.1	3466	217734	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 15 45	112	19
50-43.1a	3770	247368	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 37 21 4	113	21
50-43.1b	3770	247368	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 39 21 2 1	113	22
50-43.1c	3770	247368	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 5 25 33	113	20
51-44.1	4091	280347	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 48 12 2	114	22
52-45.1	4433	316888	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 48 9	115	22

53-46.1	4797	357292	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 14 43 6	116	23
54-47.1	5182	401924	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 21 39 3	117	24
55-48.1	5589	451125	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 27 36	118	24
56-49.1	6020	505232	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 35 28	119	25
56-49.2	6020	505233	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 35 28	119	25
57-50.1	6475	564655	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 42 21	120	26
58-51.1	6955	629798	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 48 15	121	27
59-52.1	7461	701091	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 53 10	122	28
60-53.1	7994	778988	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 57 6	123	29
61-54.1	8555	863968	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 60 3	124	30
62-55.1	9145	956536	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 62 1	125	31
63-56.1	9765	1057224	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 63	126	31
64-57.1	10416	1166592	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 63	127	32

Table 4: Generators for Table 3's (Weak) Minimum Aberration Resolution IV Designs, *n* = 128, k = 41, …, 64

Design	Generators
41-34.1a	11 13 14 19 21 26 28 31 35 38 41 49 52 56 59 61 62 67 69 73 74 76 82 84 88 97 98 100 109 110 115 117 118 122
41-34.1b	11 13 14 19 21 26 28 31 35 38 41 49 52 59 61 62 67 69 70 73 74 81 82 84 93 98 100 103 107 109 110 115 117 127
41-34.1c	11 13 14 19 21 26 28 35 38 41 49 52 56 59 61 62 67 69 70 73 74 81 82 84 93 98 100 103 107 109 110 115 117 127
41-34.1d	11 13 14 19 21 26 28 31 35 38 41 49 52 56 59 61 62 67 69 70 73 74 81 82 84 93 98 100 103 109 110 115 117 127
41-34.2a	11 14 21 22 25 26 28 35 37 38 49 50 52 55 59 61 62 67 69 74 76 79 84 87 88 93 98 100 109 110 115 117 121 124
41-34.2b	7 11 13 19 21 22 26 28 31 37 38 41 47 50 52 55 61 62 69 70 73 74 76 82 84 87 98 103 107 109 110 115 122 124
41-34.2c	13 14 22 25 26 28 35 37 38 41 42 44 52 55 61 62 67 69 70 73 76 79 81 82 84 87 94 98 103 109 112 121 124 127
41-34.2d	13 14 22 25 26 31 35 37 38 41 42 47 52 55 61 62 67 69 70 74 76 79 81 82 84 87 94 98 103 109 112 122 124 127
42-35.1a	11 13 14 19 21 26 28 31 35 38 41 49 52 56 59 61 62 67 69 70 73 74 81 82 84 93 98 100 103 107 109 110 115 117 127
42-35.1b	11 13 19 22 25 28 35 37 41 42 44 50 52 55 56 59 61 69 70 73 74 76 81 91 98 100 103 104 107 110 118 121 122 124 127
42-35.1c	13 14 16 21 22 28 31 38 41 42 47 49 50 52 56 59 61 62 67 69 70 74 76 79 81 84 87 91 97 98 100 103 110 118 121 124
43.36.1a	7 13 19 22 25 26 31 37 38 41 42 47 49 50 52 55 56 59 73 74 76 82 84 88 93 97 100 103 104 107 109 110 112 115 118 124
43-36.1b	14 21 22 25 26 28 37 38 41 42 47 52 55 56 59 61 62 69 74 76 81 82 87 91 97 98 103 104 112 115 117 118 121 122 124 127
44-37.1a	11 13 25 26 28 35 37 38 41 42 44 50 52 55 56 59 61 62 69 70 73 74 76 79 81 87 91 97 98 100 107 110 117 118 121 122 124
44-37.1b	7 11 13 19 22 25 26 31 37 38 41 42 47 49 50 52 55 56 59 73 74 76 82 84 88 93 97 100 103 104 107 109 110 112 115 118 124
44-37.1c	14 21 22 25 26 28 37 38 41 42 44 47 52 55 56 59 61 62 69 74 76 81 82 87 91 97 98 103 104 112 115 117 118 121 122 124 127
45-38.1	Same as for design 44-37.1a, plus 31
46-39.1	Same as for design 45-38.1, plus 115
47-40.1	Same as for design 46-39.1, plus 103
48-41.1	Same as for design 47-40.1, plus 19
49-42.1	Same as for design 48-41.1, plus 127
50-43.1a	7 11 13 19 22 26 31 35 37 38 41 42 44 47 49 50 52 55 56 59 73 74 76 81 82 84 88 93 94 97 98 100 103 104 107 109 110 112 115 117 118 124 127
50-43.1b	7 11 13 19 22 25 26 31 35 38 41 42 47 49 50 55 67 73 74 76 79 81 82 84 87 88 91 93 94 97 98 100 103 104 107 109 110 112 115 117 118 124 127
50-43.1c	Same as for design 49-42.1, plus 112
51-44.1	7 11 13 22 25 26 28 31 37 38 41 42 44 47 49 50 52 55 56 61 62 69 70 73 74 76 79 81 82 84 87 88 93 94 97 98 100 103 107 109 110 115 117 118
52-45.1	Same as for design 50-43.1c, plus 82 and 93

53-46.1	Same as for design 52-45.1, plus 109
54-47.1	Same as for design 53-46.1, plus 104
55-48.1	Same as for design 54-47.1, plus 88
56-49.1	19 21 22 25 26 28 35 37 38 41 42 44 49 50 52 55 56 59 61 62 67 69 70 73 74 76 81 82 84 87 88 91 93 94 97 98 100 103 104 107 109 110 115 117 118 121 122 124 127
56-49.2	Same as for design 55-48.1, plus 7
57-50.1	Same as for design 56-49.1, plus 21
58-51.1	Same as for design 57-50.1, plus 14
59-52.1	Same as for design 58.51.1, plus 22
60-53.1	Same as for design 59-52.1, plus 47
61-54.1	Same as for design 60-53.1, plus 49
62-55.1	Same as for design 61-54.1, plus 67
63-56.1	Same as for design 62-55.1, plus 84
64-57.1	Same as for design 63-56.1, plus 94. That is, 11 13 25 26 28 35 37 38 41 42 44 50 52 55 56 59 61 62 69 70 73 74 76 79 81 87 91 97 98 100 107 110 117 118 121 122 124 31 115 103 19 127 112 82 93 109 104 88 7 21 14 22 47 49 67 84 94

Table 5. A Few Good Designs That Are Not Weak Minimum Aberration

Design	w_4	df	C2FI	L	Generators
18-11.3	21	117	54	3	7, 11, 21, 45, 51, 62, 78, 86, 97, 103, 120
19-12.2	28	120	45	3	7, 11, 21, 38, 57, 76, 83, 90, 101, 111, 118, 120
20-13.2	38	123	41	4	7, 11, 21, 38, 60, 70, 73, 82, 95, 101, 107, 118, 120
21-14.8	52	127	36	5	7, 11, 13, 19, 35, 69, 70, 81, 82, 87, 98, 108, 118, 120
22-15.3	66	124	21	5	7, 11, 19, 30, 38, 41, 59, 61, 74, 85, 92, 98, 111, 118, 120
24-17.3	103	127	14	6	7, 11, 19, 29, 41, 47, 49, 59, 62, 77, 82, 92, 97, 110, 116, 119, 120

Table 6. Number of Non-Isomorphic Regular Resolution IV Designs, n = 128

k	No. of even designs, *by projection*	No. of even/odd designs, *by projection*	No. of distinct wlp for even/odd designs
12	69	180	118
13	136	487	243
14	295	1,240	448
15	596	2,926	777
16	1,292	6,208	1,278
17	2,651	11,787	1,996
18	5,598	19,466	2,890
19	11,341	27,994	4,051
20	22,728	35,192	5,211
21	43,516	39,201	6,237
22	79,603	38,847	6,546
23	?	34,868	6,361
24	?	28,133	5,656
25	?	20,569	4,709
26	?	13,498	3,575
27	?	8,075	2,611
28	?	4,284	1,720
29	?	2,149	1,119
30	?	976	632
31	?	433	340
32	?	197	177
33	?	101	90
34	?	31	30
35	?	13	13
36	?	8	8
37	?	3	3
38	?	2	2
39	?	1	1
40	?	1	1