Bayesian Modeling of Accelerated Life Tests with Random Effects

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We show how to use Bayesian modeling to analyze data from an accelerated life test where the test units come from different groups (such as batches) and the group effect is random and significant. Our approach can handle multiple random effects and several accelerating factors. However, we present our approach on the basis on an important application concerning pressure vessels wrapped in Kevlar 49 fibers where the fibers of each vessel comes from a single spool and the spool effect is random. We show how Bayesian modeling using Markov chain Monte Carlo (MCMC) methods can be used to easily answer questions of interest in accelerated life tests with random effects that are not easily answered with more traditional methods. For example, we can predict the lifetime of a pressure vessel wound with a Kevlar 49 fiber either from a spool used in the accelerated life test or from another random spool from the population of spools. We comment on the implications that this analysis has on the estimates of reliability (and safety) for the Space Shuttle, which has a system of 22 such pressure vessels. Our approach is implemented in the freely available WinBUGS software so that readers can easily apply the method to their own data.

Key Words: Markov chain Monte Carlo (MCMC), WinBUGS, Credibility Interval, Prediction Interval, Quantile.

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Introduction

The analysis of accelerated life tests is an important application in reliability. (See Meeker and Escobar, 1998 and Nelson, 1990.) Predicting the lifetime of a unit at normal operating conditions based on data collected at accelerated conditions is a common objective of these tests (Jayawardhana and Samaranayake, 2003.) In this paper we consider predicting lifetime in the case where failure time depends on random effects in addition to the accelerating stresses. We present our approach on the basis of an important application involving pressure vessels, which are critical components of the Space Shuttle, reported by Gerstle and Kunz (1983) and Feiveson and Kulkarni (2000). However, our methods are general and can be easily applied to accelerated life tests with multiple accelerating factors and random effects by straightforward modifications of the WinBUGS programs provided. (WinBUGS software is freely available at http://www.mrc-bsu.cam.ac.uk/bugs.)

Data and Initial Analyses

Gerstle and Kunz (1983) report on an accelerated life test - conducted at the U.S. Department of Energy Lawrence Livermore Laboratory (LLL) - of pressure vessels wrapped in Kevlar 49 fibers, each from a single spool (batch). The data are given in Table 1. In Table 1 are the failure times (hours) of the pressure vessels at four different fiber stresses (23.4 to 29.7 MegaPascals, by 2.1 MPa) and for eight different spools. Notice that at the lowest stress level of 23.4 MPa there were eleven right-censored observations at 41,000 hours. Gerstle and Kunz treated the spool effect as fixed and analyzed the data using analysis of variance of log failure times with standard least squares methods and a non-parametric procedure based on the Kruskal-Wallis test. Feiveson and Kulkarni (2000) in an important related paper use a version of the LLL data to assess the reliability of the pressure vessel system in the Space Shuttle, which carries 22 of these pressure vessels. Their data comes from the same accelerated life test but collected at later time and omitting the data from one of the spools because they did not consider it representative of the population of spools used in the Space Shuttle pressure vessels. We chose to use the original data reported by Gerstle and Kunz (1993) but our methods can be equally applied to Feiveson and Kulkarni's version of the data. Feiveson and Kulkarni emphasize the importance of treating the spool effect as random since the origin of the spools used in the pressure vessels of the Space Shuttle is unknown and the fibers do not necessarily come from the eight spools tested. In addition, Gerstle and Kunz (1983) report that the reason why the different spools have different failure times is unknown.

Feiveson and Kulkarni (2000) use a bootstrap based method to consider the spool effects as random. However, their method is limited in that it has trouble dealing with censored observations. It requires an assumed value of the probability of censoring at each stress. It also considers the four stress levels separately which limits the precision and accuracy of the estimates of the overall accelerated failure time model. In this paper we treat the spool effect as random using a much simpler, interpretable, applicable, general and powerful method based on Bayesian analysis using Markov chain Monte Carlo (MCMC) techniques. Our method allows one to use prior information as this is usually available especially about the shape parameter of the Weibull distribution. In addition, to the probability of failure by a given time provided by Feiveson and Kulkarni we provide credibility intervals for percentiles and prediction intervals for the failure time of new pressure vessels[1].

Prior to Feiveson and Kulkarni (2000), Crowder, et al. (1991) improved on the original analysis of Gerstle and Kunz (1983) by using a (frequentist) maximum likelihood and delta method approach to Weibull regression with fixed spool effects. They used the model to estimate percentiles of lifetime for the pressure vessels from each of the eight spools. Their results are given in Table 2. They comment that this table shows the

[1] The reader is encouraged to read the Feiveson and Kulkarni (2000) paper as it contains a brilliant analysis of the reliability of the pressure vessel system of the Space Shuttle. We believe that their analysis can be made even better if the results of this paper are combined with their results.

importance of including the spool effect in the analysis. In this table they considered two kinds of extrapolation: (a) extrapolations to very low failure probabilities, and (b) extrapolation to the presumably much longer survival times that would arise under much lower stresses than those in the experiment. In particular, they considered two specific predictions: (a) the times to which 99% of the pressure vessels will survive at stress of 23.4 MPa, and (b) the median failure time at stress of 22.5 MPa. The latter figure was chosen to represent a stress lower than any of those in the experiment but still close enough to provide reasonable extrapolations. In this paper we make similar predictions and more using the random effects model instead of the fixed effects model. See Tables 4, 5, 6 and 7 for our results. Of particular note is that we predict the life of a pressure vessel made from a random spool from the population of spools that is not one of the eight tested.

Our random effects model falls into the general category of *frailty models* in biostatistics as discussed by Ibrahim, et. al. (2001) and Congdon (2001). These models consider inference in the presence of heterogeneity in the test units such as those found with our spools. However, our results differ from those in the biomedical context, since we answer questions of interest for accelerated life tests.

Our Approach

Our primary approach to analyzing these data is to fit a Bayesian Weibull regression model with random spool effects and vague priors (Gelman, et al., 1995). However, we also consider putting informative priors on the shape parameter of the Weibull distribution since engineers typically have a sense of its value from past experience on the wear out behavior of the items being tested.

The posterior and prediction distributions involved are estimated using Markov Chain Monte Carlo (MCMC) techniques (Gamerman, 1997; Gilks, et al., 1996; Casella and George, 1992; Chib and Greenberg, 1995). Running a MCMC algorithm provides simulated values that are – after a suitable initial burn-in period – approximately distributed from the posterior distribution of the parameter of interest or the prediction distribution of a new observation[2].

WinBUGS 1.4 (Spiegelhalter, et al. 2002) provides an MCMC implementation used here. WinBUGS is downloadable for free from the WinBUGS home page at http://www.mrc-bsu.cam.ac.uk/bugs[3]. The commented code of the WinBUGS program that we use most in this paper is given in the Appendix for the reader's convenience. All other WinBUGS programs used in this paper are viewable in Adobe Acrobat PDF format and downloadable in WinBUGS format at http://web.utk.edu/~leon/bugs/. If the reader is only interested in learning the types of results that are possible with Bayesian methods using MCMC techniques there is no need to view the programs. It the reader wants to understand the mathematics behind our methods he should understand the WinBUGS code as we don't provide a separate presentation of this mathematics to avoid needless redundancy.

[2] Johnson and Albert (1999) have a concise and lucid "Review of Bayesian Computation" in Chapter 2 that is recommended for those that are new to the subject. Chapter 1 of their book "Review of Classical and Bayesian Inference" is also highly recommended.

[3] WinBUGS has a tutorial in its on-line manual that one can use to quickly learn to run the WinBUGS programs we provide. We also recommend Congdon (2001), a book full of applications with associated WinBUGS programs that can be downloaded.

The WinBUGS programs used to obtain the results in this paper available at http://web.utk.edu/~leon/bugs/ are numbered below. The numbers will be used to refer to the programs throughout this paper:

1. KevlarNoSpoolsGammaPrior.odc
2. KevlarNoSpoolsLognormalPriorI.odc
3. KevlarNoSpoolsLognormalPriorII.odc
4. Kevlar-FIXED-EffectsGammaPrior.odc
5. Kevlar-FIXED-EffectsLognormalPriorI.odc
6. Kevlar-FIXED-EffectsLognormalPriorII.odc
7. Kevlar-RANDOM-EffectsGammaPrior.odc (Code provided in the Appendix)
8. Kevlar-RANDOM-EffectsLognormalPriorI.odc
9. Kevlar-RANDOM-EffectsLognormalPriorII.odc

We ran these programs with a burn-in period of 100,000 iterations and a total of one million iterations. This number of iterations was judged sufficient to obtain samples from the posterior and prediction distributions of interest. (We examined the trace plots and used different starting values for the chains. We also ran the programs for two million iterations and there was no appreciable difference in the estimates obtained.) We remark that we had good starting values for our programs because we had maximum likelihood estimates for the parameters in the fixed effect case.

The WinBUGS programs used in this paper can be easily modified to accommodate other accelerated life test where there is (or is not) heterogeneity among groups of experimental units such as test beds, batches, etc. For example, changing the Weibull distribution assumption to lognormal involves changing just one line of code in the programs. Dealing with multiple accelerating factors is equally easy. We have used the method on the Singpurwalla, Casetellino, and Goldschen (1975) accelerated test data involving both temperature and voltage acceleration.

Fixed Spool Effect Analysis

The model considered by Crowder, et. al., (1991) in the notation of Meeker and Escobar (1998) is as follows:

Fixed Spool Effects Model:

$$F(t) = P(T \le t) = 1 - \exp\left[-\left(\frac{t}{\eta} \right)^{\beta} \right], \quad t \ge 0, \quad \beta > 0, \quad \eta > 0.$$

$$\log(\eta) = \beta_0 + \beta_1 \log(s) + \psi_k, \ \mathrm{k} = 1, \ldots 8$$

The random variable T with Weibull distribution F represents the lifetime of the pressure vessel; s is the stress level and ψ_k is the fixed spool effect where k represents the spool number. To keep the model identifiable they arbitrarily define $\psi_8 = 0$ so that $\psi_1, ..., \psi_7$ represent differences between spools 1-7 and spool 8.

Crowder, et al. (1991) main results are given in Table 2. They remark that in the case of problem (a) it is noteworthy that all the point and lower and upper confidence limits for the separate spools are much greater than the corresponding quantities for all the spools combined. The explanation is that is that the model with no spool effects leads to an estimate of the Weibull shape parameter ($\beta = 0.68$) much smaller than the estimate for the fixed spool effect model ($\beta = 1.26$). Hence, the estimated percentiles are much lower in the lower tail and much higher in the upper tail. The lesson is that ignoring a vital parameter, such as spool effect, may not only be significant in itself but may also lead to bias in estimating the other parameters.

Even though our interest is primarily in the random spools effects model not considered by Crowder, et al. (1991), for comparison purposes, we fit the fixed spools effects model using Bayesian methods with the following independent vague priors:

$$\beta_0 \sim N(0,0.001), \quad \beta_1 \sim N(0,0.001),$$
$$\beta \sim Gamma(1,0.2),$$
$$\psi_k \sim N(0,0.001), \ \mathrm{k}=1,\ldots,7$$

The normal priors for β_0 and β_1 are parameterized with the mean and the precision, τ , which is the inverse of the variance, as is commonly done by Bayesians (Congdon, 2001) and WinBUGS. The $Gamma(\alpha,\beta)$ distribution has the following density function and parameterization:

$$p(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \theta \geq 0$$
$$\alpha = Shape\ \ parameter > 0, \quad \beta = Inverse\ \ scale\ \ parameter > 0$$

The corresponding mean and variance are $E(\theta) = \frac{\alpha}{\beta}$ and $Var(\theta) = \frac{\alpha}{\beta^2}$.

The *Gamma (1, 0.2)* is used as a vague prior for the shape parameter of the Weibull because for this distribution we have:

Percent (%)	1	5	10	25	40	50	75	99
Percentile for β	0.05	0.26	0.53	1.4	2.6	3.5	6.9	23

From this table we see that this prior allows for all the likely values of the Weibull shape parameter, β, commonly found in reliability applications. (See also Figure 1 for a graph of the density *Gamma* (1, 0.2)). Recall that $\beta = 1$ corresponds to the exponential distribution with no wear out (increase in probability of failure with duration of test) and values $\beta > 1$ correspond to increasing hazard rates showing wear out. A value $\beta = 23$ corresponds to an extremely fast wear out uncommon in most reliability applications.

The results of Table 3 were obtained with Programs 1 and 4. As can be seen from this table the Bayesian estimates of the percentiles and their credibility intervals are comparable to the results obtained by Crowder, et al. (1991) given on Table 2. However, we do remark that our Bayesian estimates of the 1st percentile are smaller than those found by Crowder, et al., and this difference could be of practical interest.

Although there is close agreement between the estimates and statistical intervals of Tables 2 and 3 it should be noted than the interpretation of the statistical intervals in the two tables are completely different. The CL in Table 2 stand for *confidence limits* and 95% confidence means that on repeated random sampling 95% of the intervals so obtained will contain the true value of the parameter (percentile). The CL in Table 3 stands for *credibility limits* and 95% credibility means that we would give odds of 19 to 1 that the true value of the parameter (percentile) is contained within these particular limits.

Since engineers frequently have knowledge about the likely values of β we considered two substantially more informative lognormal priors for the shape parameter β. Similarly to Meeker and Escobar (1998 p. 348) we consider:

- **Lognormal I:** A lognormal prior for β with 0.5 percentile of 1 and 99.5 percentile of 5. Programs 2, 5 (and 8) use this prior.
- **Lognormal II:** A lognormal prior for β with 0.5 percentile of 1 and 99.5 percentile of 2. Programs 3, 6 (and 9) use this prior.

The results for the fixed effect model using these two informative priors did not differ much from the ones obtained in Table 3 using the vague prior for β. They are therefore not presented.

Random Effect Analysis

The fixed effect model inherently assumes that the eight spools of Kevlar 49 fibers comprise the entire population of interest. We believe a more realistic assumption is that these eight spools are a random sample from a larger population of spools. This assumption results in fewer parameters to be estimated in the model. This Random Spool Effects Model is given below.

Random Spool Effects Model:

$$F(t) = P(T \le t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right],\ \ t \ge 0,\ \ \beta > 0,\ \ \eta > 0.$$

$$\log(\eta) = \beta_0 + \beta_1 \log(s) + \psi_k,\ k = 1,\ldots,8$$
$$\psi_k \sim N\left(0,\ 1/\sigma^2\right),\ \mathrm{k} = 1,\ldots,8$$

The distributions of $\psi_1,...,\psi_8$ are assumed to have independent normal distributions with variance σ^2. We fit this model using Bayesian and MCMC methods implemented in WinBUGS with the following independent vague priors:

$$\beta_0 \sim N(0,0.001),\ \beta_1 \sim N(0,0.001),\ \beta \sim Gamma(1,0.2)$$
$$\tau = \left(\sigma^2\right)^{-1} \sim Gamma(0.001,0.001)$$

As in the fixed effect model the normal priors for β_0 and β_1 are parameterized with the mean and the precision. The Gamma distribution is parameterized with the shape and inverse scale parameters. The Gamma prior for the normal precision $\tau = (\sigma^2)^{-1}$ is justified as a vague prior in the Bayesian literature. See for example Congdon (2001, p. 20). Assuming a Gamma prior for the precision is equivalent to assuming an Inverse Gamma prior for the variance. A 95 % credibility interval for σ^2 is (0.7281, 7.321) indicting that the spool effect is important, as concluded by other authors mentioned in this paper.

Table 4 presents the estimates obtained for the percentiles of interest for the eight spools tested and for a random spool selected from the population of spools. In Table 5 we present prediction intervals for a new pressure vessel made from each of the eight tested spools and for a random spool. In Tables 6 and 7 we present the probability of failure by 1000 hours for a new pressure vessel from each of the eight spools and from a random spool. (For comparison purposes we include in these tables the estimates where no spool effect is considered.) All of these results were obtained with Programs 1 and 7. (Using Programs 2, 3, 8 and 9 we obtained similar results using Lognormal 1 and 2 priors for the shape parameter of the Weibull, β . These results were similar and are therefore not presented.)

It can be seen from Tables 4, 5, 6 and 7 that there is extreme uncertainty about the predicted life of a pressure vessel made from a random spool from the population of spools. This result backs the conclusion of Feiveson and Kulkarni that with the eight (seven for them) spools tested there is not enough information to be sure of the reliability and safety of the pressure vessel system in the Space Shuttle.

Comparison of the Fixed and Random Spool Effects Models

The advantage of the random spools effects model over the fixed spools effects model is that it is the correct model if the spools were selected at random from a population of spools. Most important of all is that the random effect model enables one to answer life length questions about the populations of spools -- not only about the spools selected for the accelerated life test.

Nevertheless, it is of interest to compare the fixed and random spool effect models and to determine if there is any substantial difference among the results for the *sampled spools*. (One might have used the fixed effects model even though the random spool effects model applies.) As expected, for all three priors considered for the Weibull shape parameter β , the random spool effect model as compared with the fixed effects model shrinks extreme estimates toward a middle value. This shrinking effect is seen in the one percentile estimates for extreme spools 4, 1, and 8. A similar shrinking effect for these three extreme spool numbers is seen for the fiftieth percentile estimates and for the predicted failure times of new pressure vessels.

Another method of comparison involves comparing the length of credibility and prediction intervals obtained under the fixed and random spool effect models for each sampled spool. For the three extreme spools 4, 1, and 8 the length of the credibility interval for the one and fiftieth percentiles and the prediction intervals are much wider for the fixed spool effect model than for the random spool effect model. This is the result of the shrinkage of the estimates mentioned above and the right skewness of the Weibull distribution.

One might consider the average length of these intervals (for all the eight spools) as a yardstick to make a comparison among the various models. For the gamma prior for β, when the fixed spool effects model is compared to the random spool effects model, average credibility interval length is **7.90% longer** for the estimates of 1^{st} percentile failure time at 23.4 MPa and **17.45% longer** for the estimates of 50^{th} percentile failure time at 22.5 MPa. The corresponding results for Lognormal I prior are **8.28%** and **22.46% longer**. For Lognormal II prior they are **10.85%** and **20.94% longer**.

For the gamma prior for β, when the fixed spool effects model is compared to the random spool effects model, average **prediction** (failure time) interval length is **13.55% longer** for the prediction at 23.4 MPa and **15.33% longer** for the prediction at 22.5 MPa. The corresponding results for Lognormal I prior are **12.18%** and **14.43% longer**. For the Lognormal II prior they are **15.73%** and **18.10% longer**.

Conclusion

This paper considers a very common and important problem in accelerated life testing, that is, how to make credible failure time predictions when the test units come from heterogeneous groups and the group effect is random and significant. We also show how ignoring this heterogeneity can lead to misleading and less than optimal results. We show how this problem can be solved in a straightforward fashion using random effects models, Bayesian methods and Markov chain Monte Carlo (MCMC) algorithms. We implement the approach using the freely available WinBUGS software package so that readers can easily use these methods with their data. (All versions of the WinBUGS programs used in this paper are made available to the reader on the web.) Even though we present the results on the bases of the pressure vessel data it should be noted that the programs provided can be easily modified to handle multiple acceleration factors and random effects.

Of particular note is the ability to make predictions for any group (spool) selected at random from a large population of groups (spools) and not only about the tested groups (spools). (In this application the statistical intervals obtained are wide reflecting the fact that we have a random sample of only eight spools from the population of spools.)

If the random effects model is appropriate, the random effect model shrinks estimates to a middle value for extreme groups (spools) when compared to the fixed effect model. Also the random effect model produces shorter credibility and prediction intervals than the fixed effects model for the extreme groups (spools).

Appendix : WinBUGS Code for the Random Spool Effect Model with Gamma (1, 0.2) Prior for the Weibull Shape Parameter β.

```
KEVLAR RANDOM EFFECTS GAMMA PRIOR
##########################################################################################
###############    This program represents the collaborative efforts of Avery Ashby,          ###############
###############    Ramon Leon, and Jayanth Thyagarajan. This is an inclusive odc file         ###############
###############    containing the model, data, and initial values. March 20, 2002.            ###############
##########################################################################################

##########################################################################################
###############                              MODEL                                            ###############
##########################################################################################

model KevlarRandomEffectsGammaPrior;
{

###############                    Generation of prior values                              ###############

taub ~ dgamma(0.001,0.001)                          # Precision parameter of normal random effect
sigmasquare <- 1 /  taub                            # Rescale into variance parameterization
intercept ~ dnorm(0,0.001)                          # Intercept component
beta.stress ~ dnorm(0,0.001)                        # Fixed stress effect
r ~ dgamma(1,0.2)                                   # Shape parameter of Weibull
                                                    #     inverse scale parameter that is used by
                                                    #     winBUGS parameterization for gamma

##########################################################################################

###############           This loop creates the random effect for the spools               ###############

for(i in 1:N ) {                                    # N is the number of spools (8)
  b[i] ~ dnorm(0,taub)                              # random effect of spool
  }
###############                              End of loop                                      ###############
##########################################################################################


###############    This loop reads in the data and calculates Weibull scale parameter         ###############

for(j in 1:M ) {                                    # M is the number of rows in the data (108)
  log(eta[j]) <- intercept + beta.stress * log(stress[j]) + b[spool[j]]  # This is the function for mu in the Weibull
  lambda[j] <- pow(eta[j],-r)                       # Rescale into lambda parameterization
  }                                                 #        for use in winBUGS 1.4
###############                              End of loop                                      ###############
##########################################################################################

###############    This loop contains the likelihood failure times as exact or censored      ###############

for(j in 1:M ) {
  t[j] ~ dweib(r,lambda[j])I(cen[j],)               # Failure times are Weibull or censored
  }
###############                              End of loop                                      ###############
##########################################################################################
```

```
###############  percentiles, predictions, and probabilities of failure for a given spool    ###############

for(i in 1:N) {
   eta234[i] <- exp(intercept + beta.stress * log(23.4) + b[i])        # eta values at 23.4 MPa stress for each spool
   quan234[i] <- eta234[i] * pow((-log(1 - 0.01)),(1/r))              # 1st percentile at 23.4 MPa stress for each spool
   lambda234[i] <- pow(eta234[i],-r)                                   # Rescale into lambda parameterization

   y.234new[i] ~ dweib(r,lambda234[i])                                 # Predicted distribution for 23.4 MPa stress
                                                                       #     for each spool
   probability234[i] <- 1 - exp(-(pow((1000/eta234[i]),r)))            # Prob of failure at 1000 hours for each spool

   eta225[i] <- exp(intercept + beta.stress * log(22.5) + b[i])        # eta values at 22.5 MPa stress for each spool
   quan225[i] <- (eta225[i] * pow((-log(1 - 0.5)),(1/r))) / 1000       # 50th percentile at 22.5 MPa stress for each spool
   lambda225[i] <- pow(eta225[i],-r)                                   # Rescale into lambda parameterization

   y.225new[i] ~ dweib(r,lambda225[i])                                 # Predicted distribution for 22.5 MPa stress
                                                                       #     for each spool
   probability225[i] <- 1 - exp(-(pow((1000/eta225[i]),r)))            # Prob of failure at 1000 hours for each spool
   }
###############                          End of loop                                    ###############
#############################################################################################

###############   Represents a new observation from a random spool in the population    ###############

other ~ dnorm(0,taub)                                                  # Choose a spool at random from population

eta234other <- exp(intercept + beta.stress * log(23.4) + other)  # eta values at 23.4 MPa stress for a new spool
quan234other <- eta234other * pow((-log(1 - 0.01)),(1/r))        # 1st percentile at 23.4 MPa stress for a new spool
lambda234other <- pow(eta234other,-r)                            # Rescale into lambda parameterization

y.234newother ~ dweib(r,lambda234other)                          # Predicted distribution for 23.4 MPa stress

probability234other <- 1 - exp(-(pow((1000/eta234other),r)))     # Probability of failure at 1000 hours for a new obs
                                                                 #        at 23.4 MPa

eta225other <- exp(intercept + beta.stress * log(22.5) + other)     # eta values at 22.5 MPa stress for a new spool
quan225other <- (eta225other * pow((-log(1 - 0.5)),(1/r))) / 1000   # 50th percentile at 22.5 MPa stress for new spool
lambda225other <- pow(eta225other,-r)                               # Rescale into lambda parameterization

y.225newother ~ dweib(r,lambda225other)                             # Predicted distribution for 22.5 MPa stress

probability225other <- 1 - exp(-(pow((1000/eta225other),r)))        # Probability of failure at 1000 hours for a new obs
                                                                    #        at 22.5 MPa

#############################################################################################

}
###############                       End of program model                              ###############
#############################################################################################
```

```
##########################################################################################
##############                          DATA                          ##############
##########################################################################################

list(N = 8, M = 108,
    stress = c(29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,
               29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,
               29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,29.7,
               29.7,29.7,29.7,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,
               27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,27.6,
               27.6,27.6,27.6,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,
               25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,25.5,
               25.5,25.5,25.5,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,
               23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4,23.4),

    spool  = c(2,7,7,7,7,6,7,5,2,2,2,3,5,7,3,6,3,2,7,2,7,5,8,3,2,6,2,5,4,1,8,8,1,1,1,4,
               4,4,4,3,3,3,2,3,2,2,2,1,2,6,6,2,8,1,2,4,1,6,4,4,1,8,4,6,7,3,2,2,2,2,3,8,
               2,8,2,6,8,8,6,8,8,1,5,4,1,4,1,7,7,6,3,5,2,6,5,6,5,1,1,1,1,4,4,4,4,8,8,8),

    cen = c(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
            0,0,0,0,0,0,0,0,0,0,0,0,41000,41000,41000,
            41000,41000,41000,41000,41000,41000,41000,41000),

    t = c(2.2,4,4,4.6,6.1,6.7,7.9,8.3,8.5,9.1,10.2,12.5,13.3,14,14.6,15,
          18.7,22.1,45.9,55.4,61.2,87.5,98.2,101,111.4,144,158.7,243.9,
          254.1,444.4,590.4,638.2,755.2,952.2,1108.2,1148.5,1569.3,
          1750.6,1802.1,19.1,24.3,69.8,71.2,136,199.1,403.7,432.2,453.4,
          514.1,514.2,541.6,544.9,554.2,664.5,694.1,876.7,930.4,1254.9,
          1275.6,1536.8,1755.5,2046.2,6177.5,225.2,503.6,1087.7,1134.3,
          1824.3,1920.1,2383,2442.5,2974.6,3708.9,4908.9,5556,6271.1,
          7332,7918.7,7996,9240.3,9973,11487.3,11727.1,13501.3,14032,
          29808,31008,4000,5376,7320,8616,9120,14400,16104,20231,
          20233,35880,NA,NA,NA,NA,NA,NA,NA,NA,NA,NA,NA))

##############                      End of data                      ##############
##########################################################################################

##########################################################################################
##############                    INITIAL VALUES                    ##############
##########################################################################################

list(intercept = 84.1, beta.stress = -23.1, r = 1.21, taub = 1)     # initial values based on Crowder et al. (1991)
list(intercept = 1, beta.stress = -1,r = 1, taub = 1)
##############                  End of initial values                  ##############
##########################################################################################
```

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References

Casella, G.; and George, E. I. (1992). “Explaining the Gibbs Sampler”. *American Statistician* 46, pp. 167-174.

Chib, S.; and Greenberg, E. (1995). “Understanding the Metropolis-Hastings Algorithm”. *American Statistician* 49, pp. 327-335.

Congdon, P. (2001). *Bayesian Statistical Modeling*, Wiley, New York.

Congdon P. (2003). *Applied Bayesian Modeling*, Wiley, New York

Crowder, M.J.; Kimber, A.C.; Smith, R. L.; and Sweeting, T. J. (1991). *Statistical Analysis of Reliability Data*. Chapman and Hall, London.

Feiveson, A. H.; and Kulkarni, P. M. (2000). “Reliability of Space-Shuttle Pressure Vessels with Random Batch Effects”. *Technometrics* 42, pp. 332-344.

Gamerman, D. (1997). *Markov Chain Monte Carlo*. Chapman and Hall, London.

Gelman, A.; Carlin, J.B.; Stern, H. S.; and Rubin, D.B. (1995). *Bayesian Data Analysis*. Chapman and Hall, Boca Raton, Fl, USA.

Gelfand, A. E.; and Smith, A. F. M. (1990). “Sampling-Based Approaches to Calculating Marginal Densities”. *Journal of the American Statistical Association* 85, pp. 398-409.

Gerstle, F.P.; and Kunz, S.C (1983). “Prediction of Long-term Failure in Kevlar 49 Composites” in *Long-term Behavior of Composites*, ASTM STP 813, T. K. O’Brian Ed., American Society for Testing and Materials, pp. 263-92, Philadelphia.

Gilks, W.R.; Richardson, S.; and Spiegelhalter, D.J. (1996). *Markov Chain Monte Carlo in Practice*. Chapman and Hall/CRC, Boca Raton, Fl, USA.

Hamada, M.; Pohl, A.; Spiegelman, C.; and Wendelberger, J. (2003). “A Bayesian Approach to Calibration Intervals and Properly Calibrated Tolerance Intervals”. *Journal of Quality Technology* 35, pp. 194-205.

Ibrahim, J.G.; Chen, M.H; and Sinha, D. (2001). *Bayesian Survival Analysis*. Springer-Verlag, New York. NY.

Jayawardhana, A.A.; and Samaranayake (2003). "Prediction Bounds in Accelerated Life Testing: Weibull Models with Inverse Power Law". *Journal of Quality Technology* 35, pp. 89-103.

Johnson, V.E.; and Albert J. H. (1999). *Ordinal Data Modeling*. Springer-Verlag, New York. NY.

Meeker, W.Q.; and L.A. Escobar (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analyses*. John Wiley, New York, NY.

Singpurwalla, N. D; Castellino, V. C.; and Goldschen, D. Y. (1975). "Inference from Accelerated Life Tests Using Eyring Type Re-parameterization". *Naval Research Logistics Quarterly* 22, pp. 289-296.

Spiegelhalter, D.J.; Thomas, A.; Best, N.G.; and Gilks, W.R. (2002). *WinBUGS User Manual Version 1.4*. MRC Biostatistics Unit, Institute of Public Health, Cambridge, UK and Department of Epidemiology & Public Health, Imperial College School of Medicine, London, UK. (Internet: http://www.mrc-bsu.cam.ac.uk/bugs)

Table 1: Gerstle and Kunz (1983) Data: Failure Time in hours of Kevlar 49 Wrapped Pressure Vessels

Stress	Spool	F-Time	Stress	Spool	F-Time	Stress	Spool	F-Time	Stress	Spool	F-Time
29.7	2	2.2	29.7	5	243.9	27.6	2	694.1	25.5	1	11487.3
29.7	7	4.0	29.7	4	254.1	27.6	4	876.7	25.5	5	11727.1
29.7	7	4.0	29.7	1	444.4	27.6	1	930.4	25.5	4	13501.3
29.7	7	4.6	29.7	8	590.4	27.6	6	1254.9	25.5	1	14032.0
29.7	7	6.1	29.7	8	638.2	27.6	4	1275.6	25.5	4	29808.0
29.7	6	6.7	29.7	1	755.2	27.6	4	1536.8	25.5	1	31008.0
29.7	7	7.9	29.7	1	952.2	27.6	1	1755.5	23.4	7	4000.0
29.7	5	8.3	29.7	1	1108.2	27.6	8	2046.2	23.4	7	5376.0
29.7	2	8.5	29.7	4	1148.5	27.6	4	6177.5	23.4	6	7320.0
29.7	2	9.1	29.7	4	1569.3	25.5	6	225.2	23.4	3	8616.0
29.7	2	10.2	29.7	4	1750.6	25.5	7	503.6	23.4	5	9120.0
29.7	3	12.5	29.7	4	1802.1	25.5	3	1087.7	23.4	2	14400.0
29.7	5	13.3	27.6	3	19.1	25.5	2	1134.3	23.4	6	16104.0
29.7	7	14.0	27.6	3	24.3	25.5	2	1824.3	23.4	5	20231.0
29.7	3	14.6	27.6	3	69.8	25.5	2	1920.1	23.4	6	20233.0
29.7	6	15.0	27.6	2	71.2	25.5	2	2383.0	23.4	5	35880.0
29.7	3	18.7	27.6	3	136.0	25.5	3	2442.5	23.4	1	41000.0*
29.7	2	22.1	27.6	2	199.1	25.5	8	2974.6	23.4	1	41000.0*
29.7	7	45.9	27.6	2	403.7	25.5	2	3708.9	23.4	1	41000.0*
29.7	2	55.4	27.6	2	432.2	25.5	8	4908.9	23.4	1	41000.0*
29.7	7	61.2	27.6	1	453.4	25.5	2	5556.0	23.4	4	41000.0*
29.7	5	87.5	27.6	2	514.1	25.5	6	6271.1	23.4	4	41000.0*
29.7	8	98.2	27.6	6	514.2	25.5	8	7332.0	23.4	4	41000.0*
29.7	3	101.0	27.6	6	541.6	25.5	8	7918.7	23.4	4	41000.0*
29.7	2	111.4	27.6	2	544.9	25.5	6	7996.0	23.4	8	41000.0*
29.7	6	144.0	27.6	8	554.2	25.5	8	9240.3	23.4	8	41000.0*
29.7	2	158.7	27.6	1	664.5	25.5	8	9973.0	23.4	8	41000.0*

Censored observations are indicated with an asterisk *. The stress applied to the Kevlar 49 strands in the pressure vessels are in MPa or MegaPascals.

Table 2: Frequentist Maximum Likelihood Results Reported by Crowder et al. (1991) for Fixed Spool Effect Model.

Point estimates for 1st percentile failure time at (a) 23.4 MPa; 50th percentile failure time at (b) 22.5 MPa						
Spool	(a) F-Time (hours)	Lower CL	Upper CL	(b) F-Time (x 1000 hours)	Lower CL	Upper CL
All	70	22	225	88	42	187
1	3762	1701	8317	263	138	502
2	461	222	957	32.2	19.3	54.0
3	217	95	497	15.2	8.15	28.4
4	6264	2757	14234	438	221	869
5	874	369	2070	61.1	32.0	117
6	709	322	1563	49.6	28.2	87.4
7	131	56	305	9.19	4.72	17.9
8	2108	970	4581	147	79.6	273

"All" refers to a model where no spool effect is considered. CL stands for 95% confidence limits interpreted according to the frequentist paradigm.

Table 3: Bayesian Fit of the Fixed Spool Effect Model with Gamma (1, 0.2) Prior for β.

Point estimates for 1st percentile failure time at (a) 23.4 MPa; 50th percentile failure time at (b) 22.5 MPa						
Spool	(a) F-Time (hours)	Lower CL	Upper CL	(b) F-Time (x 1000 hours)	Lower CL	Upper CL
All	62.32	17.38	177.1	73.57	40.88	135.9
1	3051	1249	6665	248.8	138.2	488.6
2	364.6	155.2	738.2	29.8	18.48	49.7
3	174	67.75	391.8	14.2	7.893	27.28
4	5015	2003	11200	409.3	219.7	825.4
5	715.7	268.6	1697	58.3	31.35	118.9
6	572.4	229.2	1243	46.7	27.22	84.9
7	104.4	40.25	238.7	8.519	4.595	16.89
8	1715	711.9	3686	140	79.94	267.2

"All" refers to a model where no spool effect is considered. The point estimates are posterior medians. The CL stands for credibility limits interpreted according to the Bayesian paradigm. These correspond respectively to the 2.5% and 97.5% percentiles of the posterior distribution of the percentiles.

Table 4. Bayesian Analysis of Random Spool Effect Model with Gamma (1, 0.2) Prior for β.

Point estimates for 1st percentile failure time at (a) 23.4 MPa; 50th percentile failure time at (b) 22.5 MPa						
Spool	(a) F-Time (hours)	Lower CL	Upper CL	(b) F-Time (x 1000 hours)	Lower CL	Upper CL
All	62.32	17.38	177.1	73.57	40.88	135.9
1	2819	1144	6117	221.5	121.1	414.6
2	362.4	153.2	732.6	28.56	17.14	46.42
3	179.5	70.06	402.3	14.08	7.616	26.86
4	4524	1773	10060	356.6	185.8	682.8
5	708.9	267.9	1657	55.42	29.92	110
6	570.5	229.3	1228	44.72	25.76	79.81
7	108.8	42.04	247.7	8.547	4.469	16.8
8	1635	675.2	3497	128.4	72.12	237.3
Random spool	671	21.96	19290	53.68	1.867	1479

"All" refers to a model where no spool effect is considered. F-time is the median of the posterior distribution of the percentile. Lower CL is the 2.5th percentile of the posterior distribution. Upper CL is the 97.5th percentile of the posterior distribution. "Random spools" corresponds to a random spool selected from the population of spools.

Table 5. Bayesian Fit of Random Effects Model with Gamma (1, 0.2) Prior for β
Prediction Intervals for the Life of a New Pressure Vessel Made from Each of the Eight Spools and from a Random Spool Selected from the Population of Spools.

(a) 23.4 MPa; (b) 22.5 MPa						
Spool	(a) F-Time (hours)	Lower CL	Upper CL	(b) F-Time (x 1000 hours)	Lower CL	Upper CL
All	29650	220	369600	73.490	0.5496	942.300
1	90820	5560	421900	218.90	13.37	1036.00
2	11750	734	51270	28.21	1.73	125.80
3	5802	354	26960	13.95	0.84	66.16
4	146200	8906	685200	350.80	21.07	1680.00
5	22950	1374	108100	55.04	3.31	264.20
6	18520	1138	83520	44.36	2.71	204.40
7	3517	211	16580	8.44	0.51	40.79
8	52910	3251	244400	126.90	7.91	598.00
Random spool	19850.00	302.70	793000	47.95	0.73	1917.00

"All" refers to a model where no spool effect is considered. The point estimates are the medians of the prediction distributions. Lower PL stands for the 2.5th percentile of the prediction distribution. Upper PL stands for the 97.5th percentile of the prediction distribution. A new pressure vessel will have a 95% probability of having a life between the lower and upper prediction limits.

Table 6. Probability of Failure by 1000 hours for a New Pressure Vessel Made from Each of the Eight Spools and from a Random Spool at 23.4 MPa Using the Random Spool Effect Model with Gamma (1, 0.2) Prior for β .

Spool	Probability of Failure (%)	Lower CL (%)	Upper CL(%)
All	0.0650%	0.0378%	0.1071%
1	0.2850%	0.0847%	0.8675%
2	3.3870%	1.5270%	7.0920%
3	7.7910%	3.3340%	16.3500%
4	0.1606%	0.0427%	0.5454%
5	1.5160%	0.5065%	4.0220%
6	1.9680%	0.7568%	4.7040%
7	13.8500%	6.1620%	27.5100%
8	0.5514%	0.1825%	1.5150%
Random Spool	1.6140%	0.0246%	60.8300%

"All" refers to a model where no spool effect is considered. The point estimates are posterior medians. The CL stands for credibility limits interpreted according to the Bayesian paradigm. These correspond respectively to the 2.5% and 97.5% percentiles of the posterior distribution of probability of failure by 1000 hours.

Table 7. Probability of Failure by 1000 hours for a New Pressure Vessel Made from Each of the Eight Spools and from a Random Spool at 22.5 MPa Using the Random Spool Effect Model with Gamma (1, 0.2) Prior for β .

Spool	Probability of Failure (%)	Lower CL (%)	Upper CL(%)
All	0.0355%	0.0184%	0.0650%
1	0.0981%	0.0246%	0.3515%
2	1.1780%	0.4467%	2.9240%
3	2.7460%	1.0150%	6.7810%
4	0.0553%	0.0123%	0.2219%
5	0.5231%	0.1509%	1.6160%
6	0.6806%	0.2234%	1.9080%
7	4.9860%	1.9210%	11.7800%
8	0.1899%	0.0534%	0.6126%
Random Spool	0.5612%	0.0080%	27.9600%

"All" refers to a model where no spool effect is considered. The point estimates are posterior medians. The CL stands for credibility limits interpreted according to the Bayesian paradigm. These correspond respectively to the 2.5% and 97.5% percentiles of the posterior distribution of probability of failure by 1000 hours.

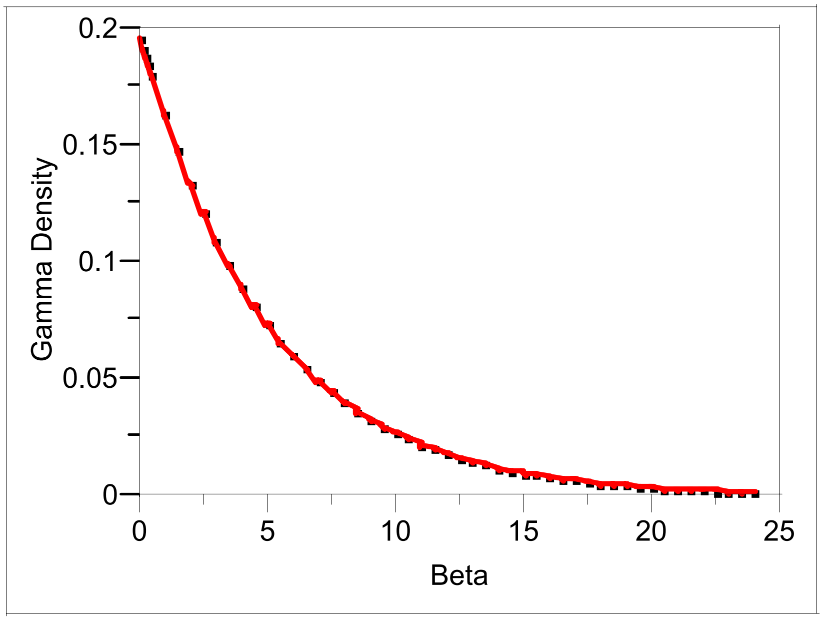

Figure 1. Density of Gamma (1, 0.2) prior