

Second Order Saturated Resolution IV Designs

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ABSTRACT

We define 2^{k-p} ($= n$) resolution IV designs which use all of their degrees of freedom to estimate main effects and two-factor interactions as second order saturated (sos) designs. We show that these designs project to every other resolution IV design. We propose a method to search for good designs using sos designs. For $k > (5/16)n$, all resolution IV designs are a projection from the even sos design at $k = n/2$. For $k \leq (5/16)n$ the minimum aberration design resolution IV designs are projections of sos designs with both even and odd words in the defining relation. While even resolution IV designs are limited to estimating fewer than $n/2$ two-factor interactions (in addition to the k main effects), resolution IV designs with odd-length words in the defining relation may devote more than half of their degrees of freedom to two-factor interactions.

Key words: alias length pattern, defining contrast subgroup, minimum aberration, projection, word length pattern.

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1. INTRODUCTION

Two-level fractional factorial designs are widely used in industrial practice to investigate the effect of factors on processes. Each factor is varied over a high and low setting of possible operating conditions to build a model to help explain the relationship of the factors to the outcome of the process. A 2^{k-p} fractional factorial design with k factors at two levels will consist of $n = 2^{k-p}$ runs. This design is a 2^{-p} th fraction of the 2^k full factorial design where the fraction is determined by p defining words. A “word” consists of letters which are the names of the factors denoted by A, B, ... or 1, 2, 3, ..., k . The number of letters in a word is the word length. The group formed by the p defining words and their generalized interactions is called the defining contrast subgroup (Wu and Hamada 2000, p.157). The defining contrast group consists of $2^p - 1$ words plus the identity column (commonly denoted as I). The defining contrast subgroup can be used to study all the aliasing relations among effects.

Every regular design can be categorized by the word length pattern of its defining contrast subgroup. For a 2^{k-p} design, let w_i denote the number of words of length i in its defining contrast subgroup. The vector $wlp = (w_1, \dots, w_k)$ is called the word length pattern of the design. The resolution of a 2^{k-p} design is defined to be the smallest r such that $w_r \geq 1$. This means the length of the shortest word defines the resolution. Box and Hunter (1961) proposed the maximum resolution criterion as a method to categorize and compare designs. Later, Fries and Hunter (1980) introduced the minimum aberration criteria. This criteria allows any two designs to be rank ordered according to their word length pattern. This is the most common criterion used today to judge the goodness of designs.

In addition to wlp, we introduce a new criterion based on the alias length pattern to help find and characterize resolution IV designs. We define the alias length pattern as the frequencies of the lengths of the alias sets for two-factor interactions:

$alp = (a_1, a_2, \dots, a_l)$ where a_1 is the number of clear two-factor interactions, a_2 is the number of pairs of aliased two-factor interactions, etc., up to a_l which is the number of the largest set of l aliased two-factor interactions $\left(l \leq \left\lfloor \frac{k}{2} \right\rfloor \right)$. The alias length pattern

(alp) also contains other important information:

- The number of degrees of freedom for two-factor interactions: $\sum_{i=1}^l a_i$
- The number of length four words in the defining relation: $w_4 = \sum_{i=2}^l \binom{i}{2} a_i / 3$.

All regular 2_{IV}^{k-p} designs of size $n = 64$ or less have been identified; see Chen, Sun and Wu (CSW) (1993) and Sun (2001). However, for $n = 128$, all possible resolution IV designs have not been identified. Indeed, the minimum aberration design is not yet reported in the design literature for some values of k . For cases with $n = 128$ or more, search algorithms are used to identify attractive fractional factorial designs having the specified size and other characteristics. For example, PROC FACTEX in SAS/QC[®] software (SAS Institute Inc., 1999) searches for minimum aberration designs for any given $k < n$. However, due to the magnitude of the computation for large n and certain values of k , exhaustive searches are not feasible given current computing speeds. This paper will propose an alternative search method for identifying good designs.

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It is well known that, for $k \leq n/2$ factors and $n = 8, 16, 24, 32, \dots$, there exist resolution IV designs. When $k = n/2$, the design is known as a *minimal design* of resolution IV (Montgomery 2001, p. 347). These minimal designs may be obtained by foldover of a saturated orthogonal main effects design of size $n/2$. For any $n = 2^r$ (with $r \geq 3$), a regular minimal design may be constructed by using all the odd interactions of the r basic columns as generators. For example, for $r = 5$, the 11 generators for the 2_{IV}^{16-11} design are the $\binom{5}{3} = 10$ three-factor interactions and the single five-factor interaction.

Alternatively one may arrange the $n-1$ columns of a saturated main effects design in Yates order (e.g., see CSW's Table 1), and:

- select every other column starting with the first or
- select the last $n/2$ columns.

Li and Mee (2002) present an alternative set of $n/2$ columns to create this minimal design. For the remainder of this article, we restrict our attention to regular resolution IV designs.

The minimal 2_{IV}^{k-p} designs are even designs, in that every word in the defining relation is of even length. Li and Mee (2002) showed that every 2_{IV}^{k-p} design with $(5/16)n < k \leq (1/2)n$ must be an even design. Even designs:

- alias even effects with other even effects, and odd effects with odd.
- allocate $n/2$ degrees of freedom to odd effects, and $n/2 - 1$ degrees of freedom to even effects
- provide at most $n/2 - 1$ degrees of freedom for estimating two-factor interactions, and at least $n/2 - k$ degrees of freedom for three-factor or higher-order interactions.

For instance, the minimum aberration 2_{IV}^{11-6} design - an even design – permits estimation of 11 main effects, 15 two-factor interactions, while leaving five degrees of freedom for aliased three-factor interactions.

By contrast, 2_{IV}^{k-p} designs with half of the words in the defining relation with odd length may provide more than $n/2 - 1$ degrees of freedom for two-factor interactions. For instance, the minimum aberration 2_{IV}^{10-5} design supports estimation of all 10 main effects and 21 two-factor interactions. Because of this greater capacity for estimating two-factor interactions, this paper will focus on the construction of even/odd 2_{IV}^{k-p} designs. While such designs do not exist for $n = 16$ and are rather rare for $n = 32$, even/odd designs are common for larger n if $k \leq (5/16)n$. Figure 1 shows the range of k/n for designs of a given size and resolution. Figure 1 shows the maximum value of k/n for regular 2_V^{k-p} designs as well as the maximum k/n for a design (not necessarily orthogonal) permitting estimation for all main effects and two-factor interactions. Note how the range of k/n increases as n increases where even/odd resolution IV designs are possible, but a higher resolution design is not available.

In the following section, we focus on the structure of even/odd resolution IV designs of size 32 and 64. We use these known cases to introduce some definitions and indicate the structure one could exploit in the larger cases where all designs are not known. In section 3, we propose a method for searching for larger even/odd designs.

2. RESOLUTION IV DESIGNS OF SIZE 32 AND 64

Only five even/odd 2_{IV}^{k-p} designs of size 32 exist; refer to Table 1. Two of these designs (10-5.1 and 9-4.2) utilize all 31 degrees of freedom for estimating main effects and two-factor interactions. We will refer to any 2_{IV}^{k-p} design (both even and even/odd designs) that use all of their degrees of freedom for estimating main effects and two-factor interactions as *second-order saturated (sos) designs*. Each of the non-sos designs is a projection of one or both of these sos designs. For instance, delete any column from design 10-5.1 and one obtains design 9-4.1.

Table 1. Even-Odd Resolution IV Designs of Size 32

<u>Design</u>	<u>Generators</u>	<u>df</u>	<u>wlp</u>	<u>alp</u>	<u>Projections</u>
10-5.1	7, 11, 19, 29, 30	31	10,16,0,0,5	0,20,0,0,1	9-4.1
9-4.1	7, 11, 29, 30	30	6,8,0,0,1	8,12,0,1	8-3.1
9-4.2	7, 11, 13, 30	31	7,7,0,0,0,1	15,0,7	8-3.1
8-3.1	7,11,29	29	3,4	13,6,1	7-2.1
7-2.1	7, 27	25	1,2	15,3	

Theorem: Every 2_{IV}^{k-p} non-sos resolution IV designs is the projection of at least one sos resolution IV parent design.

Suppose there exists a 2_{IV}^{k-p} non-sos design. A non-sos design is defined as a design that does not utilize all $2^{k-p} - 1$ degrees of freedom for estimating main effects and two-factor interactions. A non-sos design therefore has “available columns” for the unused degrees of freedom. An available column is any column that is not aliased with a main effect or two-factor interaction.

Suppose we wish to add a new factor to our design, with an available column as its generator. The new factor “ z ” multiplied by its generator will appear as an additional word in the defining contrast subgroup. The new word is necessarily of length four or more. The resulting design with $k + 1$ factors must be resolution IV for the reasons given below.

Suppose it is not resolution IV; then this would mean there is a word in the defining contrast subgroup of length three or less. This implies that a new word contains z (since z did not appear in any of the original words) plus two or fewer other factors. This implies that z is aliased with either a main effect or two-factor interaction, which contradicts the fact that the generator was an “available column”. Therefore the resulting $k + 1$ factor design must be resolution IV.

Now this $k + 1$ factor resolution IV design is either a second order saturated design with no more available columns, or a non-sos design with an available column. If not sos, the process can be repeated until the design becomes a second order saturated design. Therefore, all non-sos 2_{IV}^{k-p} designs have at least one sos parent.

Corollary: All non-sos even/odd resolution IV designs are the projection of an even/odd sos design.

Even/odd designs may project to either an even design or an even/odd design while even designs only project to other even designs (see Figure 2).

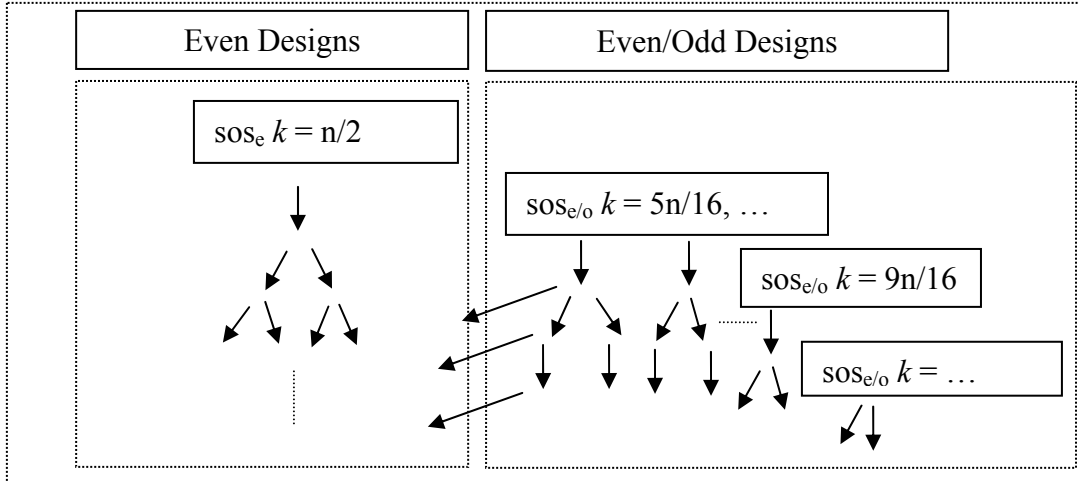


Figure 2. Schematic of Projections

Table 1 includes the generators, degrees of freedom (for main effects and two-factor interactions), word length pattern (wlp) and the alias length pattern (alp) for each of the 32-run even/odd designs. For example, design 9-4.1 has $a_1=8$ clear two-factor interactions, $a_2=12$ pairs of aliased two-factor interactions, $a_4=1$ set of four aliased two-factor interactions, and $9 + 21 = 30$ degrees of freedom for main effects and two-factor interactions.

Table 2 shows all 148 even/odd 2_{IV}^{k-p} designs of size 64. Here we use our own notation to identify the designs since CSW (1993) did not list all the designs and their ordering did not accord with any obvious criteria. We rank the alternative 2_{IV}^{k-p} designs for a given k using the following criteria:

1. Smaller w_4
2. For designs with the same w_4 , smaller w_5
3. For designs with the same (w_4, w_5) , larger a_1

To avoid confusion with the CSW numbering, we use the letters a, b, ... rather than numerals to index the designs.

We make the following observations regarding Table 2. First, there are only eight even/odd second-order saturated resolution IV designs of size 64:

- 20-14.a
- 18-12.c
- 17-11.b,d,e,g,j
- 13-7.b

Second, a non-sos design in Table 2 may be the projection of more than one sos design.

For instance, 16-10.b is the projection of either sos design 17-11.b or 17-11.d.

Note that each $n = 8, 16, 32, \dots$ there is only one even resolution IV second-order saturated (sos) design, the minimal design with $k = n/2$. Thus, the following results are apparent:

- For $n = 8$ and 16 , there exists only the unique even sos design with $k = n/2$.
- For $n = 32$, there exist three sos designs, with $k = 9, 10$, and 16 .
- For $n = 64$, there exist nine sos designs, with $k = 13, 17, 18, 20$, and 32 .

The sos designs with the smallest k are of particular interest because these designs provide the most degrees of freedom for two-factor interactions. We examine the 9-4.2 and 13-7.b designs now. Design 9-4.2 has $w_4 = 7$, and these length-four words involve only seven of the nine factors. Thus, all the interactions involving two factors are clear.

This design is structured as $\frac{1}{2} \left[2_{IV}^{7-3} \times 2^2 \right]$, where the one-half fraction of the product array is obtained by dividing each smaller design into two blocks and then taking only two of the four block combinations as follows (see Miller 1997 for this design structure):

	2^2	
	I = -59 (2 runs)	I = 59 (2 runs)
2_{IV}^{7-3} with	I = -234 (8 runs)	8x2=16 runs
	I = 234 (8 runs)	8x2=16 runs

where the 2_{IV}^{7-3} has generators 6 = 123, 7 = 124, 8 = 134. Note that the product array above is fractionated using I = +23459.

Design 13-7.b has similar structure: $\frac{1}{4} [2_{IV}^{7-3} \times 2_{IV}^{6-2}]$, with each 16-run sub-design

divided into four blocks as follows:

		2_{IV}^{6-2} with <u>11</u> = 5610 and <u>13</u> = 5612			
		<u>611</u> = + <u>61013</u> = +	<u>611</u> = + <u>61013</u> = -	<u>611</u> = - <u>61013</u> = +	<u>611</u> = - <u>61013</u> = -
2_{IV}^{7-3} with 7=123, 8=124, 9=134	1=+ 234=+	4x4 = 16 runs			
	1=+ 234=-		4x4 = 16 runs		
	1=- 234=+			4x4 = 16 runs	
	1=- 234=-				4x4 = 16 runs

3. PROPOSED DESIGN SEARCH METHOD

The difficulty in finding minimum aberration designs (and other good designs) increases dramatically as the size of the designs grows. As n becomes larger, it is no longer feasible to conduct exhaustive searches. One option is to intelligently reduce the search area. The value of sos designs is they represent a small fraction of all possible resolution IV designs and project to all the remaining possible resolution IV designs. Thus from these designs one can project to minimum aberration and other good designs.

If all the sos designs for a given n can be found and identified, then we have the starting points for all resolution IV even/odd designs for a given n .

Our proposed method to find minimum aberration and other good designs is to find all the sos designs for a given run size n and then project from those designs to identify the best designs. To accomplish this requires the ability to find sos designs, distinguish non-isomorphic sos designs, and then to find the best projections.

The first issue is no problem. It is quite easy to find sos designs. For instance, at $n = 128$ we know that sos designs exist at $k = 21, 22, 24 - 29, 31, 33, 34, 36, 40,$ and 64 . Projections of these sos designs easily lead to weak minimum aberration designs and careful evaluation of all sos designs will determine minimum aberration for any $k \leq 64$ at $n = 128$. There are over 80 unique non-isomorphic sos designs at $n = 128$ (isomorphic determined by wlp and alp).

Once all the sos designs (starting points) are known, the best set of projections must be identified to find minimum aberration designs. While this is computationally intensive it is still feasible with today's computer resources and an intelligent way to reduce the search area for good designs. Further results for $n = 128$ and larger designs will be reported elsewhere.

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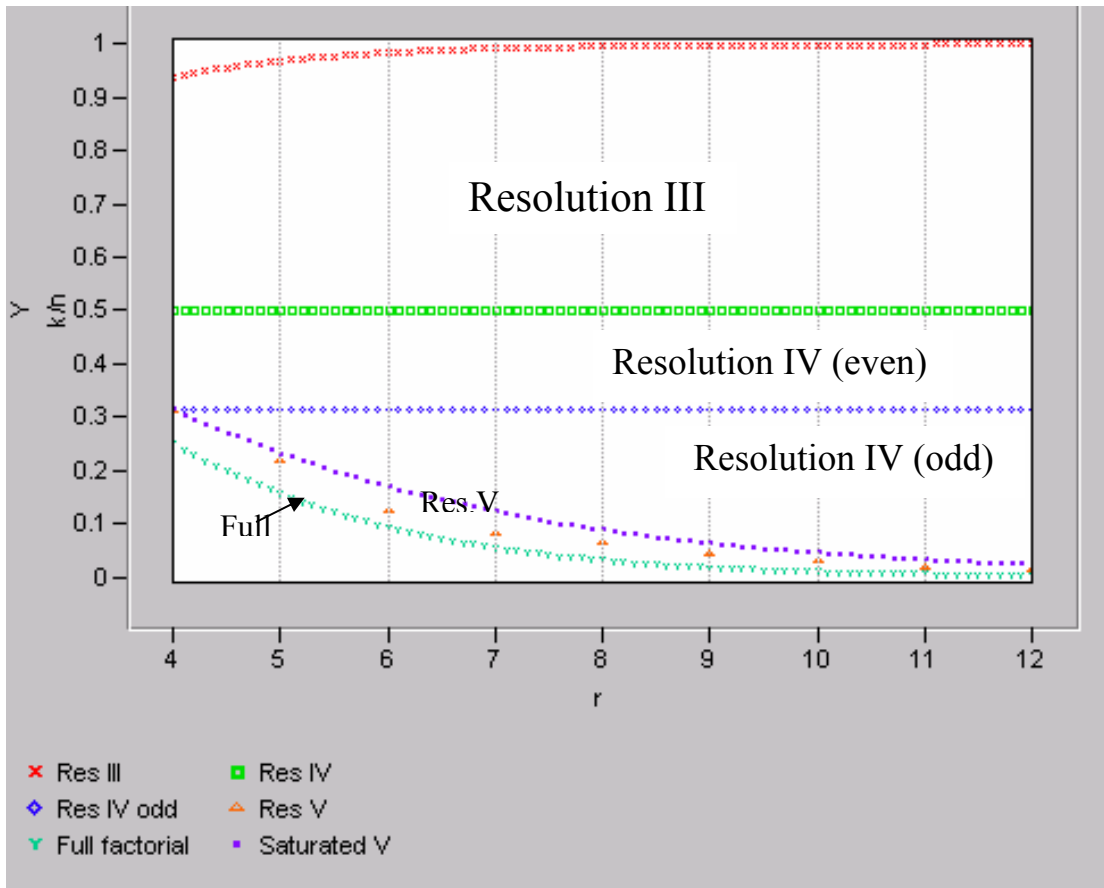


Figure 1. Resolution Graph for regular 2^{k-p} Designs of size $n = 2^r$

Table 2. Even-Odd Resolution IV Designs of Size 64

Design	Generators	d.f.	Word length pattern	Alias Length Pattern	Projections
20-14.a	7,11,13,14,19,21,22,35,37,38,57,58,60,63	63	125, 256, 480,...	0,0,0,40,0,0,0,0,3	a
19-13.a	7 11 13 14 19 21 22 35 37 38 57 58 60	62	100, 192, ...	0,0,16,24,0,0,0,3	a, b
18-12.a	7 11 13 14 19 21 22 35 37 57 58 60	61	78, 144, ...	0,3,25,12,0,0,0,3	a, c
18-12.b	7 11 13 14 19 21 22 35 37 38 57 58	61	84, 128, ...	0,16,0,24,0,0,0,2,1	c, h
18-12.c	7 11 13 14 19 21 22 25 26 35 60 63	63	92, 112, ...	0,30,0,0,0,14,0,0,1	f, i
17-11.a	7 11 13 14 19 21 35 37 57 58 60	60	59, 108, ...	0, 9, 27, 4, 0, 0, 3	a, c
17-11.b	7 11 19 29 37 41 47 49 55 59 62	63	60, 80, ...	16, 0, 0, 30	b
17-11.c	7 11 13 14 19 21 22 35 37 57 58	60	64, 96, ...	2, 14, 12, 12, 0, 0, 2, 1	c, f, g, i
17-11.d	7 11 13 19 21 25 35 37 41 49 63	63	65, 75, ...	16, 0, 15, 0, 15	b, d
17-11.e	7 11 13 14 19 21 25 35 37 41 63	63	68, 72, ...	16, 6, 0, 18, 0, 6	d, h
17-11.f	7 11 13 14 19 21 22 25 35 60 63	62	68, 88, ...	4, 26, 0, 0, 12, 2, 0, 1	e, g, j
17-11.g	7 11 13 14 19 21 22 25 35 37 63	63	73, 67, ...	19, 0, 12, 0, 12, 0, 3	h, i, j
17-11.h	7 11 13 14 19 21 22 35 37 38 57	60	76, 64, ...	16, 0, 0, 24, 0, 0, 0, 3	i,
17-11.i	7 11 13 14 19 21 22 25 26 35 60	62	84, 56, ...	16, 14, 0, 0, 0, 14, 0, 1	j, k
17-11.j	7 11 13 14 19 21 22 25 26 28 63	63	105, 35, ...	31, 0, 0, 0, 0, 0, 15	k
16-10.a	7 11 13 19 21 35 37 57 58 60	59	43, 81, ...	0, 18, 22, 0, 0, 3	a, b, d
16-10.b	7 11 19 29 37 41 47 49 55 59	61	45, 60, ...	15, 0, 15, 15	c
16-10.c	7 11 13 14 19 21 35 37 57 58	59	47, 72, ...	4, 15, 17, 4, 0, 2, 1	d, h, j, l

16-10.d	7 11 13 19 21 25 35 37 41 63	61	49, 56, ...	15, 6, 9, 9, 6	c, f, i
16-10.e	7 11 13 14 19 21 25 35 60 63	61	49, 68, ...	8, 22, 0, 9, 5, 0, 1	e, g, j, m
16-10.f	7 11 13 14 19 21 22 35 57 60	59	51, 64, ...	4, 24, 0, 12, 0, 1, 2	h, n
16-10.g	7 11 13 14 19 21 22 35 57 58	57	52, 64, ...	0, 26, 0, 12, 0, 2, 0, 1	j, n
16-10.h	7 11 13 14 19 21 25 35 37 63	61	53, 52, ...	18, 3, 9, 9, 3, 3	i, k, l, m
16-10.i	7 11 13 14 19 21 22 35 37 57	58	57, 48, ...	15, 0, 12, 12, 0, 0, 3	l, n
16-10.j	7 11 13 14 19 21 22 25 35 60	60	61, 44, ...	17, 12, 0, 0, 12, 2, 1	m, n, o
16-10.k	7 11 13 14 19 21 22 25 26 60	60	77, 28, ...	29, 0, 0, 0, 0, 14, 1	o
15-9.a	7 11 19 30 37 41 49 60 63	58	30, 60, ...	0, 30, 10, 0, 3	a
15-9.b	7 11 19 29 30 37 41 49 60	58	30, 61, ...	0, 30, 10, 0, 3	a, c
15-9.c	7 11 19 29 37 41 47 49 55	59	33, 44, ...	14, 6, 17, 7	d, g
15-9.d	7 11 13 19 21 35 37 57 58	58	33, 54, ...	6, 19, 15, 0, 2, 1	a, c, h, j, n
15-9.e	7 11 13 19 21 25 35 60 63	60	34, 52, ...	12, 18, 5, 9, 0, 1	b, e, f, j, o
15-9.f	7 11 13 19 21 35 41 49 63	59	35, 42, ...	14, 11, 8, 10, 1	d, i, k
15-9.g	7 11 13 14 19 21 41 54 63	60	35, 50, ...	12, 18, 8, 3, 3, 1	e, m, q
15-9.h	7 11 13 14 19 21 35 57 60	58	36, 48, ...	8, 20, 8, 4, 1, 2	h, l, m, r
15-9.i	7 11 13 19 21 25 35 37 63	59	37, 40, ...	17, 6, 11, 7, 3	g, i, k, n, o
15-9.j	7 11 13 14 19 21 35 57 58	56	37, 48, ...	4, 22, 8, 4, 2, 0, 1	j, m, p, r
15-9.k	7 11 13 14 19 21 35 41 63	59	39, 38, ...	19, 2, 16, 2, 4, 1	k, q, r
15-9.l	7 11 13 14 19 21 35 37 57	56	41, 36, ...	14, 3, 17, 4, 0, 3	n, r

15-9.m	7 11 13 14 19 21 25 35 60	58	43, 34, ...	18, 10, 0, 9, 5, 1	o, q, r, s
15-9.n	7 11 13 14 19 21 22 35 57	56	45, 32, ...	14, 12, 0, 12, 0, 2, 1	r, t
15-9.o	7 11 13 14 19 21 22 25 58	57	55, 22, ...	27, 0, 0, 0, 12, 3	s, t
14-8.a	7 11 19 30 37 41 49 60	57	22, 40, 36, ...	8, 26, 6, 2, 1	d, h, o
14-8.b	7 11 19 29 30 37 41 47	59	22, 40, 41, ...	16, 14, 14, 0, 1	a, i, m
14-8.c	7 11 19 29 30 37 41 49	57	22, 41, ...	8, 26, 6, 2, 1	d, i, l, o
14-8.d	7 11 19 30 37 41 52 56	57	23, 32, ...	13, 15, 12, 3	c, f
14-8.e	7 11 13 19 21 41 54 63	59	23, 38, ...	16, 17, 8, 3, 1	a, e, h, k, p
14-8.f	7 11 13 19 21 46 54 56	59	23, 40, ...	16, 17, 8, 3, 1	e, i, q
14-8.g	7 11 19 29 37 41 47 49	57	24, 31, ...	16, 9, 15, 3	f, m, o
14-8.h	7 11 13 19 21 35 57 60	57	24, 36, ...	12, 19, 9, 1, 2	d, g, k, s
14-8.i	7 11 13 19 21 41 49 63	57	25, 30, ...	16, 12, 9, 6	f, j, l, p, q
14-8.j	7 11 13 19 21 35 57 58	55	25, 36, ...	8, 21, 9, 2, 0, 1	h, i, k, r, s
14-8.k	7 11 13 19 21 35 41 63	57	26, 29, ...	18, 8, 12, 4, 1	f, j, p, s
14-8.l	7 11 13 14 19 37 57 63	57	26, 32, ...	12, 24, 0, 4, 3	g, t
14-8.m	7 11 13 14 19 35 53 57	55	27, 32, ...	8, 26, 0, 5, 1, 1	k, t, u
14-8.n	7 11 13 19 21 35 37 57	54	28, 27, ...	13, 9, 15, 0, 3	l, s
14-8.o	7 11 13 19 21 25 35 60	56	29, 26, ...	19, 8, 5, 9, 1	m, p, q, s, v
14-8.p	7 11 13 14 19 35 53 54	51	29, 32, ...	0, 30, 0, 6, 0, 0, 1	r, u
14-8.q	7 11 13 14 19 21 41 54	56	30, 25, ...	19, 8, 8, 3, 4	p, t, w, x

14-8.r	7 11 13 14 19 21 35 57	54	31, 24, ...	15, 10, 8, 4, 2, 1	s, t, u, w
14-8.s	7 11 13 14 19 21 25 54	54	38, 17, ...	25, 0, 0, 9, 6	v, w, x
14-8.t	7 11 13 14 19 21 22 57	54	39, 16, ...	25, 0, 0, 12, 0, 3	w
13-7.a	7 11 21 25 38 58 60	58	14, 28, ...	20, 18, 6, 1	b, e, g, i
13-7.b	7 11 13 30 46 49 63	63	14, 33, ...	36, 0, 14	a, h
13-7.c	7 11 19 29 37 59 62	55	15, 24, ...	12, 27, 0, 3	f
13-7.d	7 11 19 29 37 41 60	56	15, 27, ...	16, 21, 4, 2	c, g, k, m
13-7.e	7 11 13 19 46 49 63	58	15, 28, ...	22, 15, 6, 2	b, g, h, j, l
13-7.f	7 11 19 30 37 41 52	55	16, 22, ...	17, 15, 9, 1	d, f, i, m
13-7.g	7 11 13 19 37 57 63	56	16, 24, ...	18, 18, 4, 3	c, e, p
13-7.h	7 11 19 37 41 60 63	54	16, 26, ...	12, 23, 5, 0, 1	g, k, n
13-7.i	7 11 19 29 30 37 41	54	16, 28, ...	12, 23, 5, 0, 1	g, m, o
13-7.j	7 11 13 19 37 49 63	55	17, 21, ...	19, 12, 9, 2	d, i, k, l, p
13-7.k	7 11 13 19 35 53 57	54	17, 24, ...	19, 12, 9, 2	e, g, j, n, p, q
13-7.l	7 11 19 30 37 41 49	52	18, 20, 24, ...	12, 18, 6, 3	k, m
13-7.m	7 11 19 29 37 41 47	54	18, 20, 28, ...	20, 6, 14, 1	i, m, r
13-7.n	7 11 13 19 35 49 63	55	18, 21, 24, ...	21, 8, 12, 0, 1	f, l, q
13-7.o	7 11 19 29 37 41 49	52	18, 21, 24, ...	12, 18, 6, 3	m
13-7.p	7 11 13 19 21 41 54	54	19, 19, ...	20, 9, 8, 4	i, k, l, p, t, n
13-7.q	7 11 13 19 21 46 54	54	19, 20, ...	20, 9, 8, 4	l, u

13-7.r	7 11 13 19 35 53 54	50	19, 24, ...	6, 24, 6, 0, 0, 1	s, q, o, n
13-7.s	7 11 13 19 21 35 57	52	20, 18, ...	16, 11, 9, 2, 1	k, m, p, q, t
13-7.t	7 11 13 14 19 37 57	52	22, 16, ...	16, 16, 0, 5, 2	p, v
13-7.u	7 11 13 14 19 35 53	50	23, 16, ...	12, 18, 0, 6, 0, 1	q, v
13-7.v	7 11 13 19 21 25 46	51	25, 13, ...	23, 0, 5, 10	r, t, u
13-7.w	7 11 13 14 19 21 57	51	26, 12, ...	23, 0, 8, 4, 3	t, v
13-7.x	7 11 13 14 19 21 54	51	26, 13, ...	23, 0, 8, 4, 3	u, v
12-6.a	7 11 29 45 51 62	62	6, 24, ...	36, 12, 2	a
12-6.b	7 11 21 46 54 56	57	8, 20, ...	27, 15, 3	a, c, d, e
12-6.c	7 11 21 41 51 63	55	9, 18, ...	24, 15, 4	c, h
12-6.d	7 11 21 41 54 56	53	10, 15, ...	21, 15, 5	b, d, h
12-6.e	7 11 19 37 57 63	53	10, 16, 12, ...	20, 18, 2, 1	c, g, j
12-6.f	7 11 19 29 37 59	53	10, 16, 16, ...	20, 18, 2, 1	d, f, l
12-6.g	7 11 19 29 37 57	53	10, 18, ...	20, 18, 2, 1	c, e, h, i, j, l
12-6.h	7 11 13 30 46 49	56	10, 20, ...	30, 6, 8	a, e, k, m
12-6.i	7 11 21 25 38 58	52	11, 14, ...	21, 12, 7	d, g, h, o
12-6.j	7 11 19 37 57 60	51	11, 16, ...	16, 21, 0, 2	e, j
12-6.k	7 11 19 37 41 60	50	12, 13, ...	17, 15, 5, 1	h, j, n, o
12-6.l	7 11 13 19 46 49	52	12, 14, 12, ...	23, 9, 7, 1	d, h, j, k, p, r
12-6.m	7 11 19 29 37 41	50	12, 14, 12, ...	17, 15, 5, 1	h, l, o

12-6.n	7 11 19 37 57 58	49	12, 16, ...	12, 23, 1, 0, 1	i, j, r
12-6.o	7 11 19 29 30 37	49	12, 20, ...	12, 23, 1, 0, 1	i, l, s
12-6.p	7 11 13 19 37 57	50	13, 12, ...	19, 12, 5, 2	g, h, j, p
12-6.q	7 11 13 19 35 53	48	14, 12, ...	15, 14, 6, 0, 1	j, l, p, q
12-6.r	7 11 21 25 31 45	48	15, 10, ...	21, 0, 15	o
12-6.s	7 11 19 29 30 35	43	15, 16, ...	0, 30, 0, 0, 0, 1	q, s
12-6.t	7 11 13 19 21 57	48	16, 9, ...	21, 3, 9, 3	n, o, p
12-6.u	7 11 13 19 21 46	48	16, 10, ...	21, 3, 9, 3	r, p, o
12-6.v	7 11 13 14 19 53	48	18, 8, ...	21, 8, 0, 6, 1	p, t
11-5.a	7 11 29 45 51	55	4, 14, ...	34, 9, 1	a, c, f
11-5.b	7 25 42 52 63	51	5, 10, ...	25, 15	b
11-5.c	7 11 29 46 49	52	5, 12, ...	28, 12, 1	a, b, d, e
11-5.d	7 11 21 46 56	50	6, 10, ...	25, 12, 2	b, c, e, h
11-5.e	7 11 29 45 49	50	6, 12, 4, ...	25, 12, 2	a, e, f
11-5.f	7 11 19 29 62	51	6, 12, 8, ...	27, 12, 0, 1	c, j
11-5.g	7 11 21 38 57	48	7, 8, ...	22, 12, 3	b, g
11-5.h	7 11 21 41 51	48	7, 9, ...	22, 12, 3	b, e, g, h
11-5.i	7 11 19 29 45	48	7, 12, ...	21, 15, 0, 1	d, e, f, i, j
11-5.j	7 11 19 37 57	46	8, 8, ...	18, 15, 1, 1	e, g, i
11-5.k	7 11 13 30 49	49	8, 10, 4, ...	28, 3, 7	c, e, k, l

11-5.l	7 11 19 29 37	46	8, 10, 4, ...	18, 15, 1, 1	e, h, j
11-5.m	7 11 13 30 46	49	8, 14, ...	28, 3, 7	f, l
11-5.n	7 11 21 25 63	45	9, 6, ...	19, 9, 6	g
11-5.o	7 11 21 25 45	45	9, 7, ...	19, 9, 6	g, h
11-5.p	7 11 13 19 53	45	10, 6, ...	21, 6, 6, 1	g, h, i, k
11-5.q	7 11 19 29 35	42	10, 8, 0, ...	10, 20, 0, 0, 1	i, j
11-5.r	7 11 13 19 46	45	10, 8, 4, ...	21, 6, 6, 1	j
11-5.s	7 11 19 29 30	42	10, 16, ...	10, 20, 0, 0, 1	k
11-5.t	7 11 13 14 51	45	14, 4, ...	27, 0, 0, 7	h, i, l
10-4.a	7 27 43 53	49	2, 8, ...	33, 6	a, c
10-4.b	7 25 42 53	46	3, 6, ...	27, 9	a, c
10-4.c	7 11 29 51	47	3, 7, ...	30, 6, 1	a, e, f
10-4.d	7 11 29 46	47	3, 8, ...	30, 6, 1	a, f
10-4.e	7 11 29 49	44	4, 6, ...	24, 9, 1	a, b, c, d, f
10-4.f	7 11 29 45	44	4, 8, ...	24, 9, 1	c, f
10-4.g	7 11 21 57	42	5, 4, ...	21, 9, 2	b, d
10-4.h	7 11 21 45	42	5, 5, ...	21, 9, 2	b, d, e, f
10-4.i	7 11 19 45	40	6, 4, ...	17, 12, 0, 1	d, f
10-4.j	7 11 19 29	40	6, 8, ...	17, 12, 0, 1	f
10-4.k	7 11 13 51	41	7, 3, ...	24, 0, 7	d, e

10-4.1	7 11 13 30	41	7, 7, ...	24, 0, 7	f
9-3.a	7 27 45	42	1, 4, ...	30, 3	a, c
9-3.b	7 25 43	39	2, 3, ...	24, 6	a, b, c
9-3.c	7 27 43	39	2, 4, ...	24, 6	c
9-3.d	7 11 53	37	3, 2, ...	21, 6, 1	b, c
9-3.e	7 11 51	37	3, 3, ...	21, 6, 1	c
9-3.f	7 11 29	37	3, 4, ...	21, 6, 1	c
8-2.a	15 51	36	0, 2, 1, ...	28	-
8-2.b	7 57	33	1, 1, ...	22, 3	-
8-2.c	7 27	33	1, 2, ...	22, 3	-