Some New Second-Order Designs

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Abstract

New second-order designs are proposed which can be constructed by augmenting an initial 2^{k-p} fractional factorial of resolution IV or lower. Two classes of designs are proposed. One class is similar to central composite designs, except the factorial portion is an irregular – and generally non-orthogonal – fraction. The second class of designs proposed is similar to Box and Behnken's (1960) three-level designs in that each design combines two-level factorials in subsets of the factors. However, by utilizing an initial block that is a regular two-level fractional factorial design, our designs are both smaller and more efficient. Numerous designs are proposed for both types. Using variance dispersion graphs as well as the estimation efficiency for individual parameters, we compare the new designs with a wide variety of second-order designs from the literature.

Key words: A-efficiency, central composite design, second-order model, sequential experimentation, small composite design, three-level designs, variance dispersion graph.

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Numerous designs have been proposed for estimating the coefficients of a second-order polynomial model in *k* factors. Some designs such as those proposed by Box and Behnken (1960) and Morris (2000) require only three values for each factor. Alternatively, when the experimental region is spherical, there exist central composite (CC) designs that are essentially optimal (Myers and Montgomery, 1995, p. 370).

Sequential assembly of designs has been a part of response surface methodology ever since the seminal work of Box and Wilson (1951). The fact that CC designs are constructed by adding an axial point block to commonly-used two-level factorials contributes largely to the popularity of CC designs. Frequently, however, the experimenter recognizes the need for a second-order design only after the factorial portion is completed. When the number of factors is small, e.g., $k \leq 4$, typical two-level designs commonly used are either full factorials or half-fractions which can be augmented with the other half. In either case, the initial screening design is one block of the standard CC design. Similarly, when k = 5, the common 2_V^{5-1} screening design is part of the standard CCD. However, for k > 5, initial fractional factorial designs are generally resolution III or IV, designs not suited for augmentation to a usual CC design. For instance, for seven factors, the standard CC design requires a 2^{7-1} fraction with resolution V or higher. Any orthogonal 64-run design constructed by augmenting an initial 2_{IV}^{7-3} design is still resolution IV. Thus, the standard CC design cannot utilize these 16 runs. The purpose of this article is to propose more economical second-order designs for a moderate or large number of factors – designs that can be constructed from an initial resolution IV (or lower) fractional factorial. We now introduce our two approaches to constructing such designs.

Repaired resolution composite designs

Consider the case with k = 9 factors, based on an initial 2_{IV}^{9-3} with generators 7 = 123, 8 = 1245, and 9 = 1346. This design may be run in four blocks of size 16 [blocking on 156 and 234, each block corresponds to Chen, Sun, and Wu's (1993) design 9-5.2] or even eight blocks of size 8. This 64-run design has only one length-four word in the defining relation, I = 1237 = ..., which aliases three pairs of two-factor interactions. To this standard 2_{IV}^{9-3} design, we propose adding another block of eight runs defined as follows:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
-1.5	-1.5	-1.5	0	0	0	1.5	0	0
-1.5	-1.5	1.5	0	0	0	-1.5	0	0
-1.5	1.5	-1.5	0	0	0	-1.5	0	0
-1.5	1.5	1.5	0	0	0	1.5	0	0
1.5	-1.5	-1.5	0	0	0	-1.5	0	0
1.5	-1.5	1.5	0	0	0	1.5	0	0
1.5	1.5	-1.5	0	0	0	1.5	0	0
1.5	1.5	1.5	0	0	0	-1.5	0	0

Note that this block is a 2_{IV}^{4-1} fractional factorial with generator 7 = -123, rescaled so that the levels of the factor settings are $\pm\frac{\sqrt{k}}{\sqrt{4}}$ = 1.5 for factors 1, 2, 3, and 7, with all other factors set to zero. Note that by rescaling to $\pm\frac{\sqrt{k}}{\sqrt{4}}$, these treatment combinations are the same distance from the center as were the original 2_{IV}^{9-3} runs. With the addition of these eight runs, all two-factor interactions are estimable. Hence, if we now add, e.g., six centerpoint replicates, as well as 18 axial runs with $\alpha = \sqrt{k}$ = 3, we have an efficient, 96-run second-order design for spherical regions.

We will refer to such designs as *repaired resolution central composite* (RRCC) designs. Additional examples for $6 \le k \le 12$ are proposed in a subsequent section. The example just presented, however, illustrates the potential gain in run efficiency. Whereas a standard CC design for $k = 9$ requires 128 factorial points, this RRCC requires only 72 (=64+8) factorial runs. In fact, this completed RRCC design is substantially smaller than the 2_V^{9-2} fraction by itself. Although not as frugal as the small composite design proposed by Draper and Lin (1990), the RRCC design's estimation efficiency is much better (see Sec. 3).

New sequential three-level designs

We now propose another class of second-order designs obtained by augmenting an initial 2^{k-p} screening design of resolution III or IV. We introduce the approach with the following example. Suppose experimentation begins with the 2_{IV}^{7-3} fractional factorial with generators 5=234, 6=134 and 7=124. Next, we add seven different 2_{IV}^{4-1} fractional factorials, each with the remaining three factors set to zero. The seven 2_{IV}^{4-1} designs are each based on the negative of one of the seven words in the defining relation of the original 2_{IV}^{7-3}. The [16 + 7(8) =] 72-run design is shown in the table below:

Block	1	2	3	4	5	6	7	Design generators	
1	±1	±1	±1	±1	±1	±1	±1	2_{IV}^{7-3} w/ 5=234, 6=134, 7=124	16-run initial fraction with 2 centerpoint replicates
	0	0	0	0	0	0	0		
2	±1	±1	0	0	±1	±1	0	2_{IV}^{4-1} w/ 6 = -125	56 Augmenting Runs
	±1	±1	0	±1	0	0	±1	2_{IV}^{4-1} w/ 7 = -124	
	±1	0	±1	±1	0	±1	0	2_{IV}^{4-1} w/ 6 = -134	
	±1	0	±1	0	±1	0	±1	2_{IV}^{4-1} w/ 7 = -135	
	0	±1	±1	±1	±1	0	0	2_{IV}^{4-1} w/ 5 = -234	
	0	±1	±1	0	0	±1	±1	2_{IV}^{4-1} w/ 7 = -236	
	0	0	0	±1	±1	±1	±1	2_{IV}^{4-1} w/ 7 = -456	

We will refer to designs of this type as *sequential three-level* (S3L) designs. The A-efficiency over the hypercube for this new design is 32%, double the A-efficiency of 16% for the Box-Behnken (BB) three-level design. While the total size of this S3L design is twelve runs larger than the BB design, following an initial 2_{IV}^{7-3}, it is more economical to complete the S3L design than it is to run the BB design. Additional S3L designs for k = 6, 8, and 9 are proposed in a later section.

The remainder of this paper is organized as follows. The next section presents the class of repaired resolution central composite designs. This is followed by a brief section on sequential three-level designs. The paper concludes with a summary of other second-order designs and an evaluation of our new designs vis-a-vis the alternatives. Throughout this paper, we assume a second-order model of the form:

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij} x_i x_j + \varepsilon$$

where the *k* variables $x_1, \ldots, x_k$ are coded with initial factorial levels ±1, there are $p = .5(k+1)(k+2)$ unknown coefficients $\beta_0, \beta_1, ..., \beta_k, \beta_{11}, ..., \beta_{kk}, \beta_{12}, \beta_{13} ..., \beta_{k-1,k}$, and the errors ε are independently distributed with mean 0 and variance σ^2.

1. REPAIRED RESOLUTION CENTRAL COMPOSITE (RRCC) DESIGNS

The standard CC designs require resolution V or more in the factorial portion. For *k* > 5, the usual initial screening designs are resolution III or IV, and these are not easily augmented to resolution V. Furthermore, orthogonal resolution V designs can be quite large. Consider the case where *k* = 7 factors. There are only 28 effects to be estimated, yet the smallest orthogonal 2_V^{k-p} has 64 runs, a degree of freedom efficiency of less than 50%. In the past, the excessive run size for CC designs has been addressed by turning to small central composite (SCC) designs which are based on resolution III* or irregular Plackett-Burman designs for the factorial portion (Hartley 1959, Draper 1985, Draper and Lin 1990). While SCC designs are much smaller than CC designs, they have very poor estimation efficiency (Myers and Montgomery 1995). Furthermore, many SCC designs are based on fractional factorials that would not be recommended as an initial screening designs by itself. By contrast, we have sought to develop RRCC designs that i) utilize common screening designs, and ii) provide good estimation efficiency. While the resulting designs generally have seven levels - as did the k = 9 factor example presented earlier - they are easy to construct conceptually. Our RRCC designs are similar in all respects to the standard CCD, except the CC design's larger orthogonal factorial portion is replaced by a 2^{k-p} (of resolution III or IV) repaired with small fractional factorial designs in subset(s) of the factors rescaled to lie the same distance from the center as the original factorial treatment combinations. If the repairing fraction is based on a subset of s factors (s = 3 or 4), then the rescaled levels are $\pm\sqrt{k}/\sqrt{s}$.

We now present a second RRCC design. Consider the case where k = 7: a 2_{IV}^{7-2} fractional factorial (32-run) design with the generators 6=123, and 7=1245 and word length pattern = (0, 1, 2, 0). This initial design may be run in two (resolution III) 16-run blocks, blocking on 134. To remove the aliasing that results from the length-four word 1236, add one 2^{4-1} block with generator 6 = -123, rescaled so that the levels of the factor settings are $\pm\sqrt{7}/\sqrt{4}$ for columns 1, 2, 3, and 6 (all other factors are set to zero). Finally, add a block with 14 axial runs (with $\alpha=\sqrt{7}$) and six centerpoint replicates for a final design size of 60 runs. The standard CC design requires 40% more runs.

Eleven different RRCC designs are proposed in Table 1. The design above is Design 7b. An alternative RRCC design is proposed for k = 7 which requires four fewer runs. Both this smaller RRCC design and the one for k = 6 in Table 1 are based on 16-run resolution III* fractions rather than the better initial design of resolution IV. These two RRCC designs are equivalent to Hartley's (1959) SCC design augmented with four rescaled factorial runs to improve the precision of the design.

For k = 9, …, 12, the proposed RRCC designs are based on the minimum aberration 64-run fractions. No RRCC design is proposed for larger k for the following reasons. For k = 12, …, 15, there exists an irregular 128-run orthogonal design that supports estimation of all main effects and two-factor interactions (see Hedayat, Sloane, and Stufken 1999). Note that the smallest regular resolution V fractions are twice this size. If a composite second-order design were desired for these values of k, a CC design would best be constructed using this run-efficient orthogonal 128 design.

Table 1: Repaired Resolution Central Composite (RRCC) Designs

k	Initial Factorial	Repairing Fractions	Size
6	2^{6-2}_{III*} w/generators: 5=12, 6=34	Two 2^{3-1}_{III} , I =-125, I =-346	(16 + 8) + 12 + n_c
7	<u>Design 7a</u> 2^{7-2}_{III*} w/generators: 6=12, 7=1345	One 2^{3-1}_{III} , I =-126	(32 + 4) + 14 + n_c
7	<u>Design 7b</u> 2^{7-2}_{IV} w/ generators: 6=123, 7=1245	One 2^{4-1}_{IV} , I =-1236	(32 + 8) + 14 + n_c
8	2^{8-3}_{III} w/ generators: 6=12, 7=13, 8=2345	Two 2^{3-1}_{III} , I =-126, I =-137, One 2^{4-1}_{IV} , I =-2367	(32 + 16) + 16 + n_c
8b	2^{8-3}_{IV} w/ generators: 6=123, 7=124, 8=1345	Three 2^{4-1}_{IV} , I =-1236, I =-1247, I =-3467	(32 + 24) + 16 + n_c
9	2^{9-3}_{IV} w/ generators: 7=123, 8=1245, 9=1346	One 2^{4-1}_{IV} , I =-1237	(64 + 8) + 18 + n_c
10	2^{10-4}_{IV} w/ generators: 7=123, 8=1245, 9=1246, $\underline{0}$=1356	Two 2^{4-1}_{IV} , I =-1289, I =-4567	(64 + 16) + 20 + n_c
11	2^{11-5}_{IV} w/ generators: 7=123, 8=124, 9=1345, $\underline{0}$=1346, $\underline{1}$=1256	Four 2^{4-1}_{IV} , I =-129$\underline{0}$, I =-3467, I =-4567, I =-3568	(64 + 24) + 22 + n_c
12	2^{12-6}_{IV} w/ generators: 7=123, 8=124, 9=1345, $\underline{0}$=1346, $\underline{1}$=1256, $\underline{2}$=23456	Six 2^{4-1}_{IV} , I =-4567, I =-3478, I =-3568, I =-29$\underline{12}$, I =-129$\underline{0}$, I =-1$\underline{012}$	(64 + 48) + 24 + n_c

2. SEQUENTIAL THREE-LEVEL (S3L) DESIGNS

Three-level second-order designs have been popular, especially for $k = 3$ and 4. Box and Behnken (BB) (1960) constructed three-level designs based on combining 2^s or 2^{s-1} designs in subsets of s factors. The subsets were determined using the structure of (partially) balanced incomplete block designs. Mee (2001) presents additional Box-Behnken-type designs for k = 4, 5, …, 21 factors that are more run efficient than previous BB designs. Rechtshaffner (1967), Webb (1971), Hoke (1974), Mitchell and Bayne (1976, 1978), and Notz (1982) all propose small three-level designs. However, none of these designs are constructed by augmenting a regular 2^{k-p} design. The only three-level design class of which we are aware that augments an initial 2_{III}^{k-p} design is Morris's (2000) *augmented pair* designs. However, Morris (2000) did not investigate the use of blocking for his designs. Our S3L designs are proposed as a means of following an initial 2^{k-p} fractional factorial with additional a second block of runs to complete an efficient second-order design with only three levels.

A S3L design for k = 7 factors in 74 runs was presented earlier. Consider now $k = 8$. Following an initial 2_{IV}^{8-4} design augmented with centerpoint runs, the investigator recognizes the need for a second-order design. No suitable BB design exists. (For $k = 8$, Box and Behnken (1958) list a 245-run design, which is clearly larger than necessary, having more than five times as many runs as parameters to be estimated). For a S3L design, one takes each of the 14 length-four words in the defining relation for the 2_{IV}^{8-4} design and constructs a 2_{IV}^{4-1} design in that subset of factors. [These $14(2^{4-1}) = 112$ runs actually form half of the BB design mentioned above.] The complete S3L design is:

Block	1	2	3	4	5	6	7	8		
1	±1	±1	±1	±1	±1	±1	±1	±1	2_{IV}^{8-4} w/ 5=234, 6=134, 7=124, 8=123	Initial Design 16 + n_0 Runs
	0	0	0	0	0	0	0	0	centerpoint runs	
2	±1	±1	±1	0	0	0	0	±1	2_{IV}^{4-1} w/ 8 = -123	112 Augmenting Runs
	±1	±1	0	±1	0	0	±1	0	2_{IV}^{4-1} w/ 7 = -124	
	±1	±1	0	0	±1	±1	0	0	2_{IV}^{4-1} w/ 6 = -125	
	±1	0	±1	±1	0	±1	0	0	2_{IV}^{4-1} w/ 6 = -134	
	±1	0	±1	0	±1	0	±1	0	2_{IV}^{4-1} w/ 7 = -135	
	±1	0	0	±1	±1	0	0	±1	2_{IV}^{4-1} w/ 8 = -145	
	±1	0	0	0	0	±1	±1	±1	2_{IV}^{4-1} w/ 8 = -167	
	0	±1	±1	±1	±1	0	0	0	2_{IV}^{4-1} w/ 5 = -234	
	0	±1	±1	0	0	±1	±1	0	2_{IV}^{4-1} w/ 7 = -236	
	0	±1	0	±1	0	±1	0	±1	2_{IV}^{4-1} w/ 8 = -246	
	0	±1	0	0	±1	0	±1	±1	2_{IV}^{4-1} w/ 8 = -257	
	0	0	±1	±1	0	0	±1	±1	2_{IV}^{4-1} w/ 8 = -347	
	0	0	±1	0	±1	±1	0	±1	2_{IV}^{4-1} w/ 8 = -356	
	0	0	0	±1	±1	±1	±1	0	2_{IV}^{4-1} w/ 7 = -456	

The full S3L design for *k* = 8 is still rather large. Morris' augmented pair design requires only 83 runs. However, if following the initial 16-run design, one were to determine that a single factor could be dropped from further experimentation, then only seven of the augmenting sets of runs would be required. For example, dropping factor 1, one may eliminate the first half of Block 2 and the resulting 16+7(8) run design is the S3L design for *k* = 7.

Note that dropping a factor from the *k* = 7 S3L design does not produce a second-order design. Instead, for k = 6, we propose a S3L design based on an initial 2_{III}^{6-3} design. The defining relation for this design has 4 length-three words and 3 length-four words. Thus, the proposed S3L design augments the 8-run initial design with four 2_{III}^{3-1} fractions and three 2_{IV}^{4-1} fractions (refer to Table 2). This design is rather similar to the augmenting pair design for k = 6. While Morris's (2000) design augments the 2_{III}^{6-3} design with 33 additional runs, our S3L design requires $4(2^{3-1}) + 3(2^{4-1}) = 40$ more runs. The efficiencies of these two designs are compared in the next section.

Table 2: Sequential Three-Level Designs

k	**Initial Factorial**	**Augmenting 2_{s}^{s-1} fractions**	**Run size***
6	2_{III}^{6-3} design (4=123, 5=23, 6=13)	Three 2_{IV}^{4-1} fractions with defining words -1234, -1256, and -3456 Four 2_{III}^{3-1} fractions with defining words -136, -145, -235, and -246	$(8+n_0) + 40$
7	2_{IV}^{7-3} design (5=234, 6=134, 7=124)	Seven 2_{IV}^{4-1} fractions w/defining words -1247, -1256, -1346, -1357, -2345, -2367, and -4567.	$(16+n_0) + 56$
8	2_{IV}^{8-4} design (5=234, 6=134, 7=124, 8 =123)	Fourteen 2_{IV}^{4-1} fractions w/defining words -1238, -1247, -1256, -1346, -1357, -1458, -1678, -2345, -2367, -2468, -2578, -3478, -3568, and -4567.	$(16+n_0) + 112$
9	2_{IV}^{9-4} design (6=123, 7=124, 8=135, 9=145)	Nine 2_{IV}^{4-1} fractions w/defining words -1236, -1247, -1358, -1459, -2568, -2579, -3467, -3489, and -6789.	$(32+n_0) + 72$

* For recommendations on the number of centerpoint replicates, see Section 3.

Finally, we consider the case for *k* = 9. The S3L design proposed is based on an initial 2_{IV}^{9-4} design with nine length-four words (see Table 2) – a design that can be run in two blocks of size 16. We choose this initial design, even though there are two 2_{IV}^{9-4} designs with fewer length-four words, because one needs factorials in at least *k* different subsets of factors in order to estimate the *k* pure quadratic coefficients. The A-efficiency over the hypercube for this 104-run S3L design is 25%, while the 130-run BB design has A-efficiency of only 9%.

3. COMPARISON OF SECOND-ORDER DESIGNS

Table 3 lists a number of second-order designs as well as their run sizes for $2 \le k \le 12$. The first five columns of the table may be used in a spherical region (Small CCD through Box-Behnken-type), since all the points except for centerpoint runs fall on the same radius - with radius $\sqrt{k}$ for the composite designs and $\sqrt{s}$ for the BB designs with $s < k$. (This assumes that $\alpha = \sqrt{k}$ for the axial points of the composite designs.) Columns four through seven are three-level designs (Box-Behnken through Small 3-Level Designs). Central composite designs with $\alpha = 1$ provide another alternative when restricted to three levels per factor. Each of the three-level designs are appropriate for hypercube experimental regions – with the possible exception of the BB (-type) designs. (BB designs provide inferior precision for main effects and two-factor interactions since they do not place any treatment combinations in the corners of the hypercube.)

Although Table 3 contains a vast array of second-order designs, it is certainly not exhaustive. Second-order designs that appeared subsequent to Myers, Khuri, and Carter's review of response surface methodology (1989) include:

- optimal designs that can accommodate most run sizes for any factor by Hardin and Sloane (1993),
- simplex-shell designs and other response surface designs by Crosier (1991 and 1993), and
- 3-level factorial designs using partially balanced arrays by Katsounis (2000).

Giovannitti-Jensen and Myers (1989) proposed using variance dispersion graphs (VDGs) as a succinct evaluation of a design's prediction variance properties. Myers, et al. (1992) used the VDG to compare a number of second-order response surface designs.

Appendix 1(2) contains VDGs for repaired resolution central composite designs (sequential three-level designs). In each case, the VDGs contrast our new design with an alternative design from the literature.

We provide a brief summary of the results displayed in the Appendices. The prediction variance efficiency is best for the CC and RRCC designs, and worst for the small composite designs – although the small composite designs require the fewest runs. We consider our RRCC designs to be the best compromise between run size (not too large to be feasible) and variance efficiency (much better than the small composite designs).

Among three-level designs, our designs have superior efficiency. For k = 6, the S3L design is more nearly rotatable than the BB design, Morris's augmented pair design, or the small three-level design by Notz. For k = 7 and 8, the S3L design is compared with a BB design, and for k = 9, the S3L is compared with a design from Hoke (1974). In each case, the sequential three-level design is superior.

The variance dispersion graphs in Appendices 1 and 2 have been done for models without a term for block differences. We are currently working on recommendations for the number of centerpoint replicates for each design. Preliminary conclusions indicate that for RRCC designs, it is best to place the centerpoint runs in the axial block, while for S3L designs, the centerpoint runs are more helpful if they are included in the initial factorial block. Consistent with Draper (1982), relatively few centerpoint replicates are needed to achieve adequate precision with the new second-order designs we propose.

We particularly encourage the use of the S3L designs following an initial 2^{k-p} design that fills the experimental region of interest. Also promising are the RRCC designs following a 2_{IV}^{k-p} fraction with few length-four words, or as an improvement to Hartley's (1959) small composite designs. An electronic version of Table 3 with hyperlinks to each design listed may be found at http://stat.bus.utk.edu/techrpts/2nd_orderdesigns_files/2nd_orderdesigns.htm. Further study of these designs in continuing, and any reader's feedback is welcomed.

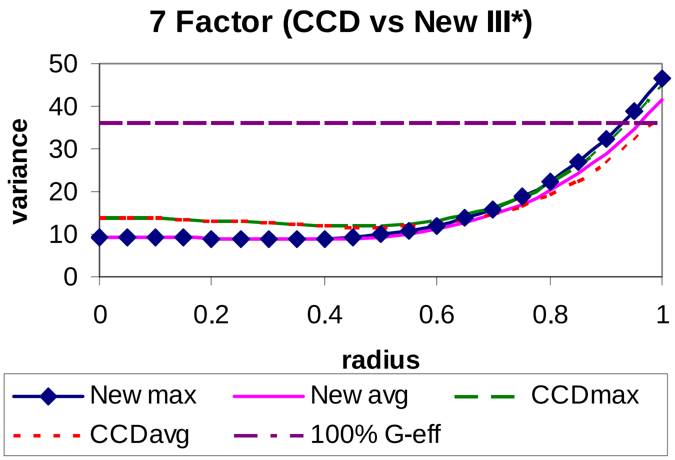
Table 3. Catalog of Alternative Second-Order Designs

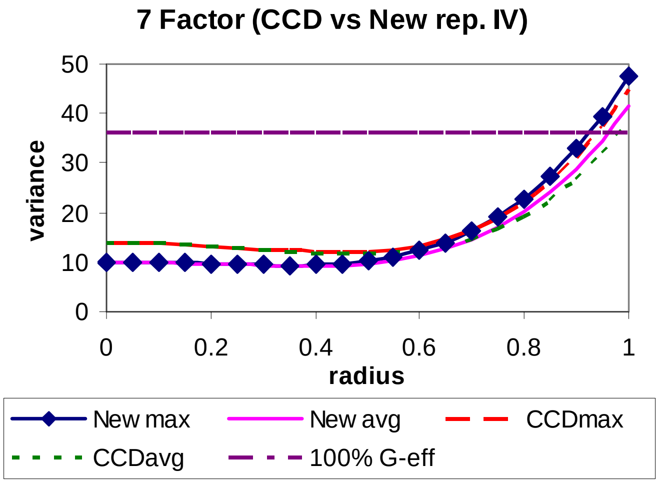
Spherical Designs							
				Three-Level Designs			
k	**Small CC**	**New Repaired Resolution CC**	**Standard CC**	**Box-Behnken**	**BB-Type (Mee 2000)**	**New Sequential 3-level**	**Small 3-level**
2	Hartley – 6 + n_c		8 + n_c				Notz – 6 (Recht.) Mitchell/Bayne – 6
3	Draper/Lin –10 + n_c Hybrid – 10 + n_c		14 + n_c	12 + n_c			Notz –10 (Recht.) Mitchell/Bayne – 10 Hoke – 13 Webb – 14 Katsaounis – 10
4	Draper/Lin –16 + n_c Hybrid – 16+ n_c John's ¾ - 20+ n_c		24 + n_c	24 + n_c	32 + n_c		Notz – 15 (Recht.) Mitchell/Bayne – 15 Dubova/Federov –15 Hoke – 19 Katsaounis – 15
5	Westlake – 22 + n_c Draper/Lin - 21+ n_c		26 + n_c	40 + n_c	new B-B 40 + n_c		Notz – 21 Mitchell/Bayne – 21 Rechtschaffner – 21 Hoke – 26 Katsaounis – 21

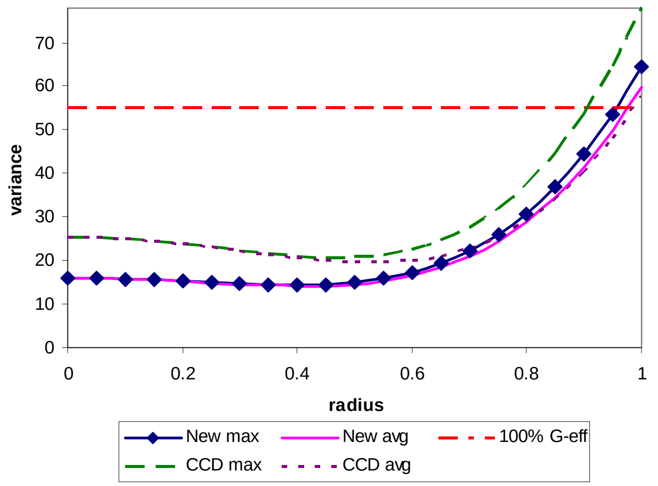
Table 3. Catalog of Alternative Second-Order Designs (continued)

Spherical Designs							
				Three-Level Designs			
k	**Small CC**	**New Repaired Resolution CC**	**Standard CC**	**Box-Behnken**	**BB-Type (Mee 2000)**	**New Sequential 3-level**	**Small 3-level**
6	Draper/Lin – 28 + n_c Hybrid – 27+ n_c	2^{6-2}_{III*} *repaired* 36 + n_c	44 + n_c	48 + n_c	96 + n_c	2^{6-3}_{III} *augmented* 48+ n_c runs	Notz – 28 Hoke – 34 Rechtschaffner – 28 Augmented Pair – 41 Katsaounis – 28
7	Draper/Lin – 36 + n_c Hybrid – 45+ n_c John's ¾ – 62+ n_c Recht. 43+ n_c	2^{7-2}_{IV} *repaired* 54 + n_c 2^{7-2}_{III*} *repaired* 50 + n_c	78 + n_c	56 + n_c	new B-B 56 + n_c	2^{7-3}_{IV} *augmented* 72+ n_c runs	Rechtschaffner – 36 Augmented Pair – 41 Hoke – 43 Katsaounis – 36
8	Draper/Lin – 46 + n_c John's ¾ 64+ n_c	2^{8-3}_{IV} *repaired* 72 + n_c 2^{8-3}_{III} *repaired* 64 + n_c	80 + n_c	192 + n_c	128 + n_c new B-B 112+8	2^{8-4}_{IV} *augmented* 128+ n_c runs	Rechtschaffner – 45 Hoke – 53
9	Draper/Lin – 56 + n_c Mee – 82 + n_c	2^{9-3}_{IV} *repaired* 90 + n_c	146+ n_c	120 + n_c	96 + n_c 144 + n_c	2^{9-4}_{IV} *augmented* 104+ n_c runs	Rechtschaffner – 55 Hoke – 64
10	Mee 84+ n_c Draper/Lin - 66+ n_c	2^{10-4}_{IV} *repaired* 100+ n_c	148 + n_c	160 + n_c	160 + n_c		Rechtschaffner – 66 Hoke – 76 Augmented Pair – 83
11	John's ¾ 118+ n_c	2^{11-5}_{IV} *repaired* 118+ n_c	150+ n_c	176 + n_c			Augmented Pair – 83
12	OA – 152 + n_c	2^{12-6}_{IV} *repaired* 136+ n_c	280+ n_c	192 + n_c	192 + n_c		

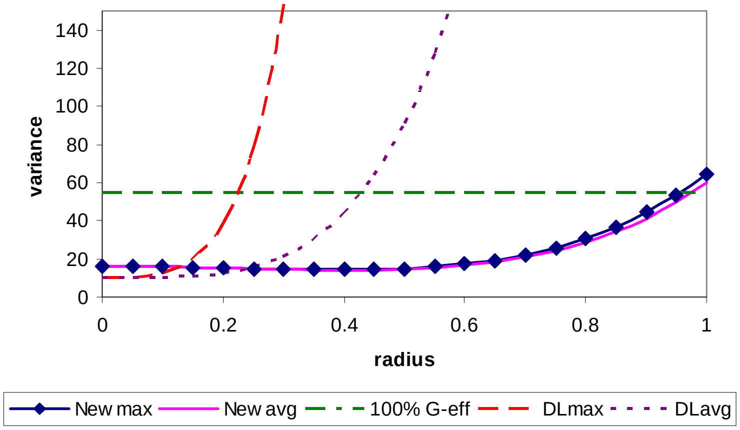
Appendix 1:
Variance dispersion graphs for new repaired resolution central composite designs



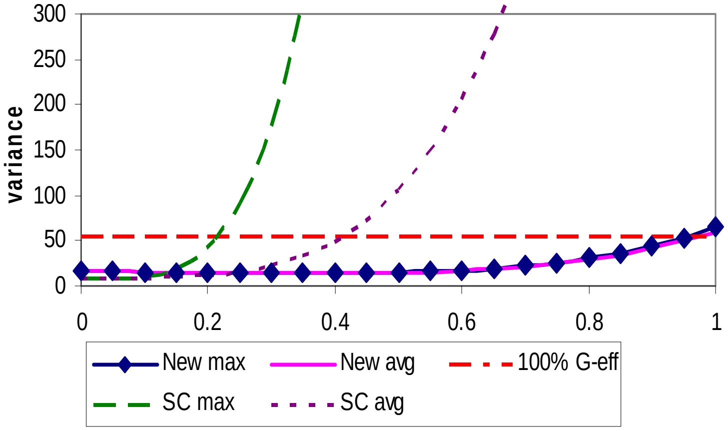




9 Factor (CCD vs New)



9 Factor, Draper/Lin vs rep CCD



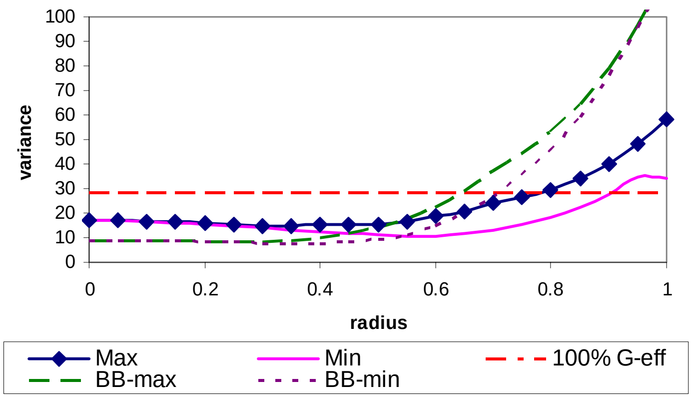
9 Factor (Small CCD vs New)

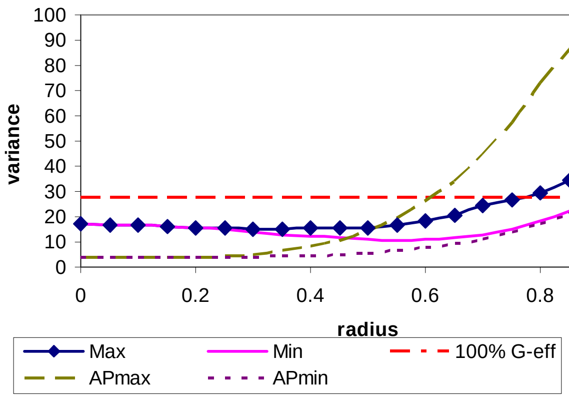
Appendix 2: Variance dispersion graphs and parameter estimation efficiency for new sequential three-level designs

The results presented below are for the new S3L designs without centerpoint runs and without blocking. When an additive block effect is included in the model, one must include centerpoint runs, preferably in the initial block.

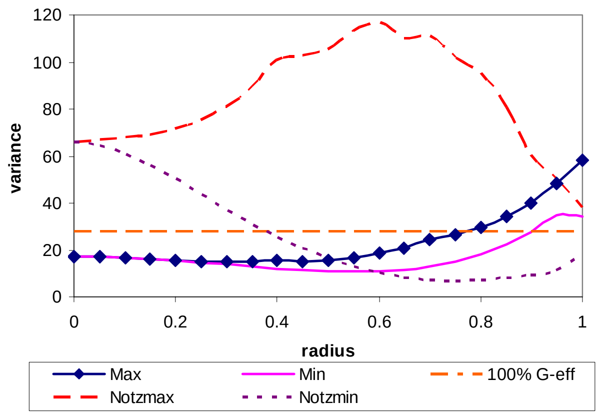
k = 6 S3L Design

	Var($\hat{\beta}$)	Efficiency
Main effects	$\sigma^2/30.4$	.633
Bi-Linear effects	$\sigma^2/19.5$ or $\sigma^2/24$	.41-.5
Quadratic effects	$\sigma^2/9.6$	.2


6 Factor, 3-Level (Box-Behn. vs New)


6 Factor, 3-Level (Aug. Pair vs New)

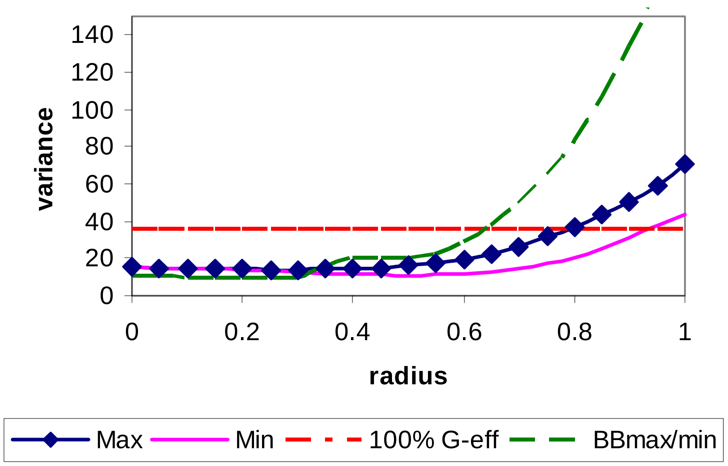
6 Factor, 3-Level (Notz vs New)



k = 7 S3L Design

	Var($\hat{\beta}$)	Efficiency
Main effects	$\sigma^2 / 48$	.667
Bi-Linear effects	$\sigma^2 / 28.8$	.399
Quadratic effects	$\sigma^2 / 16$	.222

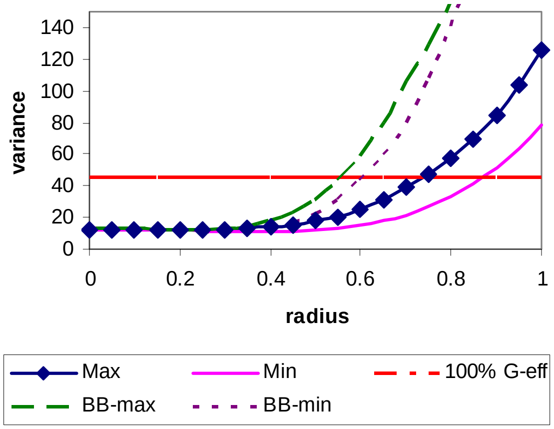
7 Factor, 3-Level (BB vs New)



k = 8 S3L Design

	Var($\hat{\beta}$)	Efficiency
Main effects	$\sigma^2/72$	.563
Bi-Linear effects	$\sigma^2/36.6$	.285
Quadratic effects	$\sigma^2/31.4$	.245

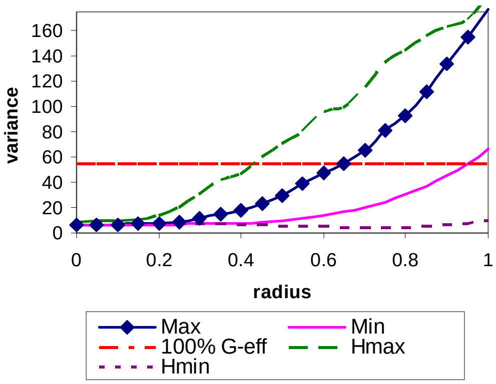
8 Factor, 3-Level (BB vs New)



k = 9 S3L Design

	Var($\hat{\beta}$)	Efficiency
Main effects	$\sigma^2/64$	.615
Bi-Linear effects	$\sigma^2/25.6$ or $\sigma^2/32$	.246-.307
Quadratic effects	$\sigma^2/14$	.135

9 Factor, 3-Level (Hoke vs New)



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