FRACTIONAL FACTORIAL DESIGNS THAT RESTRICT THE NUMBERS OF TREATMENT COMBINATIONS FOR FACTOR SUBSETS

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SUMMARY

Designs are constructed that limit the number of treatment combinations within subsets of factors. Such designs came to our attention in the context of machine assembly experiments [1,2]. For example, to improve a stepper motor, we plan a 2^{4-1} design to investigate four rotor factors and a similar 2^{4-1} design to investigate four stator factors. The product of these two arrays - to study the eight factors of these motors - would involve 64 combinations. However, a resolution IV design is possible in only 16 treatment combinations. Others have cited the benefits of such design efficiencies for assembly experiments. We propose two methods for constructing restricted subset designs and catalog the possible designs with factors partitioned into two or more subsets. We also suggest alternative applications of these designs and cite related literature.

KEY WORDS: aliasing, experimental design, linear graph, prototype experiments

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INTRODUCTION

In many experiments with a large number of factors, the factors naturally correspond to distinct groups. The following four examples in recent literature illustrate this feature:

- Bisgaard [2] presents an eight-factor assembly experiment with Mercury Marine engines, used to determine tolerances for throttle handle components. The factor subsets correspond to three components of the throttle: three knob factors, three handle factors and two tube factors.
- Miller [3] presents an experiment investigating how to reduce wrinkling in laundered clothes for an appliance manufacturer in New Zealand. Miller's strip-plot experiment studied six washer factors and four dryer factors.
- Lopez-Alvarez and Aguirre-Torres [4] present a series of experiments for an automotive paint supplier. These experiments were motivated by complaints of white paint yellowing on flexible molded parts. The first experiment reported by Lopez-Alvarez and Aguirre-Torres involved ten factors, including six clear coat factors and two oven factors. The third experiment they report involved five base coat factors, four clear coat factors and a single oven factor.
- Sonius and Tew [5] discuss designed experiments for a computer model to increase strength of a metal end fitting's connection to a composite tube. Nine factors were studied: four factors defining the interface surface between the composite and metal fitting, four factors related to the fitting geometry, and a single factor related to the length of the thickened end portion of the composite tube.

For each of these experiments, the factors may be partitioned into two or three subsets of factors. Furthermore, the amount of work to complete such physical experiments often depends not only on the number of treatment combinations in the entire experiment, but also on the number of treatment combinations for each factor subset. Bisgaard [2] emphasizes how experiment costs depend not only on the overall number of runs but also on the number of different prototype handles and knobs required - though this was seen clearly only following the experiment (see Section 6, p. 149). In Miller's example [3], the costs associated with the number of washer and dryer loads - as distinct from the number of combined washer/dryer treatment combinations - was recognized prior to the experiment and incorporated into the design choice.

For the Lopez-Alvarez and Aguirre-Torres examples [4], each distinct treatment combination for the clear coat factors required a separate batch of paint to be prepared. Furthermore, since only a small portion of each clear coat batch was used, more distinct batches implied both more preparation time and greater material costs. The first paint experiment involved $2^{6-1} = 32$ different clear coat batches, with each batch being used only once. Since the 2^{10-5} design used was resolution IV, a different assignment of the ten factors to columns of the design could have required only half as much paint by using only $2^{6-2} = 16$ different clear coat batches. For the other Lopez-Alvarez and Aguirre-Torres example mentioned above, we find in a later section that similar savings were possible.

In the context of experiments involving assembly of components, both Diamond [1] and Bisgaard [2] discuss designs that limit the number of treatment combinations for subsets of factors. Diamond refers to such designs restricting the number of subset treatment combinations as Smoak-modified designs, in recognition of Marvin Smoak's work developing a stepper motor. Diamond [1, pp. 130-131] illustrates a simple algorithm for determining two subsets of four factors for Smoak's 2^{8-4} stepper motor application, and then lists six 32-run or larger examples with up to four factor subsets, with varying numbers of factors per subset.

Bisgaard [2] presented a two-step process for constructing such designs, which we summarize here. Given that the factors under study have been appropriately subsetted, the first step is to determine the design to be used for each subset of factors. Bisgaard [2] refers to this as *pre-fractionation*. Pre-fractionation determines the number of treatment combinations per subset. The product array is then formed from the designs for individual subsets. The second step in the design process is *post-fractionation*. Post-fractionation uses one or more contrasts to pare down the product array to a manageable fraction.

The next section of this article presents lists of 16-run and 32-run designs for which all the factors are assigned to subsets that restrict the number of treatment combinations. Refer to Tables 1 and 2. A subsequent section critiques the construction methods suggested by Diamond [1] and Bisgaard [2], and describes two other means of constructing designs that restrict the numbers of treatment combinations for factor subsets. One of the new construction methods utilizes linear graphs. The other construction method we recommend utilizes the chains of aliased main effects and two factor interactions. These construction methods may be employed if the desired design is not found in Table 1 or 2. The article concludes with several sequential experimentation applications, and a few summary remarks.

DESIGNS WITH ALL FACTORS IN RESTRICTED SUBSETS

Tables 1 - 2 appearing on the following pages list restricted subset designs for 16 and 32 runs, respectively. Designs are presented for up to 16 factors, with two or more subsets of factors. Each design listed meets the following criteria:

- all $k \leq 16$ factors are assigned to a subset
- each subset of factors has a restricted number of treatment combinations. That is, it must have fewer treatment combinations than

  i) a full factorial in those factors, and

  ii) the total number of runs for the design

Restriction i) implies that the minimum subset size is 3 (4) for Resolution III (IV) designs. Restriction ii) implies a maximum subset size of 7 for the 16-run Resolution III designs in Table 1, and a maximum of 8 factors per subset for the 32-run Resolution IV designs in Table 2. Designs where one or more factors are not in restricted subsets are more easily constructed than designs with all factors so restricted. Refer to the following section for means of construction for cases not listed in these tables.

To obtain Table 1, all possible 16-run non-isomorphic designs were studied. These designs were obtained from [6]. All designs were studied with only one design given for a particular partition. For Table 2, all possible 32-run resolution IV designs were studied for k = 8, 9, …, 16.

Usage of the tables is illustrated by an example with ten factors, in two subsets of five factors each, in a 16-run design. Let A - E denote the first subset of factors and let M - Q denote the second set of factors. Referring to Table 1, we find that a resolution III design exists where each subset of factors requires eight treatment combinations. Table 1 also indicates how one should assign each subset of factors to an orthogonal array, namely that columns 1-5 should be assigned to A-E and columns (6,8,9,14,15) should be assigned to M-Q. Referring to Table 3, we see that columns 1, 2, 4, and 8 are independent columns and the other columns are defined as products of these. Here, we assign any three factors from the first subset (e.g., A-C) to columns 1, 2, and 4, respectively, and one factor from the second subset (e.g., M) to column 8. The remaining factors are defined as follows:

$$D = 3 = 1*2 = A*B$$

$$E = 5 = 1*4 = A*C$$

$$N = 6 = 2*3 = B*C$$

$$O = 9 = 1*8 = A*M$$

$$P = 14 = 2*4*8 = B*C*M$$

$$Q = 15 = 1*2*4*8 = A*B*C*M.$$

These generators provide a 1/4th fraction of the $2^{5-2} \times 2^{5-2}$ product array. The vast majority of other assignments of factors to the ten columns of this resolution III design would require 16 treatment combinations for each of the subsets.

ALTERNATIVE METHODS FOR CONSTRUCTING RESTRICTED SUBSET DESIGNS

As far as we know, constructing designs that limit the number of treatment combinations within subsets of factors has not been discussed in the literature beyond what appears in [1,2]. Here we illustrate how such designs can also be found using either linear graphs or alias relationships. We also show how defining relation “word length patterns” can be used to determine which fractional factorial designs cannot be obtained by post-fractionation of a product array.

To illustrate the usefulness of each method, we use the third stage situation reported by Lopez-Alvarez and Aguirre-Torres [4]. That experiment involved 10 two-level factors which naturally partitioned into one subset of five base coat factors and a second subset of four clear coat factors; the tenth factor was oven temperature. The authors reported that a 17-run design was obtained using ECHIP™ software in order to estimate the 10 main effects. That design required 13 distinct base coat combinations and 11 distinct clear coat combinations. We note that the usual 2_{III}^{10-6} design with one fewer run has advantages of orthogonality for the main effects and the possibility of requiring only eight clear coat combinations and eight base coat combinations. We now consider alternative means of identifying such a design, labeling the factors as A, B_1-B_5, and C_1-C_4 for the oven, base coat, and clear coat factors, respectively.

Attempted construction based on methods of Diamond and Bisgaard

We have not found Diamond's suggested approach based on finding pairs of rows that are similar to be useful. For the 10-factor paint application, we begin with the 2^{10-6} design offered by JMP (see Table 4). This is the minimum aberration design, i.e., the design with the fewest length three words. We recognize quickly that each pair of rows (1-2, 3-4, etc.) is identical for the first three columns and the tenth column. Thus we consider assigning columns X1-X3 and X10 to the four clear coat factors. Now we must find which five of the remaining six columns can be assigned to the base coat factors. However, no matter which of six columns X4-X9 we eliminate, there are 16 treatment combinations (not 8) in the remaining five factors.

Thus, no solution exists with columns X1-X3 and X10 assigned to the subset of four factors. Although one might be able to find a suitable design by considering other subsets for the first group of four columns, Diamond's method gives no assistance in selecting such a subset. We despair of using this method and proceed to the method proposed by Bisgaard [2].

Bisgaard's pre-fractionation step is obvious here. Since we desire eight treatment combinations for the clear coat factors, we select the 2^{4-1} design defined by $I = C_1C_2C_3C_4$. For eight treatment combinations in the five basecoat factors we select the quarter fraction $I = B_1B_2B_3 = B_3B_4B_5 = B_1B_2\,B_4B_5$. The product array of these two designs and the simple 2^1 design for A is the 2^{10-3} fraction with defining contrast $(I = C_1C_2C_3C_4) \times (I = B_1B_2B_3 = B_3B_4B_5 = B_1B_2\,B_4B_5)$. To obtain a sixteen-run design, we must find a one-eighth fraction of this design. That is, we must post-fractionate this product array using three linearly independent contrasts. The fact that such a design exists is encouraged by the fact that the word length pattern for the product array is (2,2,0,0,2,1) which is consistent with the word length pattern for the minimum aberration 2^{10-6} design, (8,18,16,8,8,5). However, finding three independent contrasts that result in only eight length three words - and no shorter words - is not a simple task.

We now propose two methods for finding a solution to this design problem that most practitioners will find simpler. The first is based on the use of linear graphs [7]. The second method is based on examining the alias relationships for the 2^{10-6} design.

Using Linear Graphs to Recognize Factor Subsets

Designs with restricted subsets can be obtained using Taguchi's linear graphs [7]. Figure 1 shows linear graphs for selected designs of size 4 and 8. Familiarity with the Figure 1 graphs enable one to recognize subsets of factors corresponding to disjoint portions of the linear graphs for the L_{16} or larger arrays. Figure 2 shows five linear graphs corresponding to five 2^{15-11} restricted subset designs. The numbers 1-15 in the linear graphs correspond to the 15 linearly independent columns as identified in Table 3. The notation (4,5,6) for the last graph in Figure 2 denotes a design with subsets of four, five, and six factors, respectively. These particular graphs are identical to ones given by Taguchi [7, p. 1131].

To identify a design with one subset of four factors and another of five factors, we could simply use the (4,5,6) design as follows: assign B_1-B_5 to the group of five factors with columns 2, 3, 4, 5, and 6; assign C_1 -C_4 to the group of four factors with columns 10, 11, 12, and 13; assign A to any one of the six remaining columns, e.g., column 1. This design satisfies the criterion of restricting the number of base coat and clear coat combinations to eight each. However, this design happens to have nine words of length three rather than eight. From [6] we know that we would obtain the design with only eight length-three words if we used the four basic columns 1, 2, 4, 8, plus columns 3, 5, 6, 9, 14, and 15. Looking back at the (4,5,6) linear graph in Figure 2, the minimum aberration design is obtained if we choose Column 1 for A, the group of five factors for B_1 -B_5, and Columns 8, 9, 14, and 15 for C_1 -C_4. For additional comments about the use of linear graphs, see the Appendix.

Using Aliasing Structure

The ten-factor paint example has its simplest solution using this method. We begin by constructing the minimum aberration design given in Table 4. Examining the aliased main effects and two factor interactions, we find two main effects that are aliased with four two-factor interactions, e.g., X5 = X1*X6 = X2*X7 = X3*X9 = X4*X10. The first three aliases in this chain (X5 = X1*X6 = X2*X7) coincide with a 2^{5-2}_{III} design, while the remaining aliases (X3*X9 = X4*X10) correspond to the 2^{4-1}_{IV} design. Thus by assigning B_1 -B_5 to X1, X2, X5, X6, X7, C_1 -C_4 to X3, X4, X9, X10, and A to the remaining factor X8, we achieve the desired design.

Using chains of aliased effects provides a simple means of constructing good restricted subset designs. As in our example, one begins with the minimum aberration design (or some reasonable design), and searches the alias relation for subsets of factors corresponding to the desired individual array.

Recommendations regarding construction methods

Now we highlight the strengths and weaknesses we see in each of these methods for constructing restricted subset fractional factorial designs. Diamond's method is the least useful and is not discussed further. The pre-/post-fractionation method of Bisgaard is intuitive; it is also quite useful if the post-fractionation step is not difficult, e.g., if one only needs half of the product array. However, in the ten-factor example we considered, finding an acceptable 1/8th of the product array was problematic. For such cases where the final design is much smaller than the product array of the restricted subsets, linear graphs are recommended to those not able to generate the aliasing structure of candidate designs. By using these graphs in conjunction with a good design taken from [6], one can identify restricted subset possibilities. However, for one able to generate and interpret aliasing patterns for any specified design, the chains of aliased main effects and lower-order interactions are most useful for identifying restricted subset designs of resolution III or IV.

SEQUENTIAL EXPERIMENTATION WITH RESTRICTED SUBSETS

Here we consider design options for k = 15 factors and k = 18 factors, respectively. These examples are included to illustrate using linear graphs and alias relationships, in addition to Tables 1-2, to identify attractive designs with factor subsets. We also explore the possibility of augmenting an initial design with a second design the same size. Finally, we call attention to the use of designs first highlighted by Li and Mee [8] for constructing restricted subset designs with resolution IV subsets.

Fifteen factor example with three subsets of five factors: several alternatives

Table 1 offers a $(2_{III}^{5-2} \times 2_{III}^{5-2} \times 2_{III}^{5-2}) / 2^5 = 2_{III}^{15-11}$ design. This initial design is displayed in the first part of Figure 3, where *a1-a8* denote the first subset treatment combinations, while *b1-b8* and *c1-c8* denote the second and third subset combinations. Since this is a saturated main effects design, providing no degrees of freedom for interactions, we consider increasing the number of treatment combinations to 32. Table 2 offers a $(2_{IV}^{5-1} \times 2_{IV}^{5-1} \times 2_{IV}^{5-1})/2^7 = 2_{IV}^{15-10}$ design with 15 degrees of freedom for main effects and 15 degrees of freedom for two-factor interactions. This design can be obtained by folding over the initial 16-run design. This follow-up design would be analogous to the initial design, except that each of the subset treatment combinations (*a1-a8, b1-b8, c1-c8*) would be replaced by its mirror image. For example a1 =

(-,-,-,+,+) becomes a1' = (+,+,+,-,-), a new subset treatment combination. Hence, this pair of 16-run designs is somewhat unsatisfactory in that it requires twice the effort of the 16-run design; it doubles not only the total number of runs but also the number of subset combinations. As an alternative, one could begin with the 16-run design (with subsets as defined in Figure 2), and follow that with a second fraction that requires only eight new subset treatment combinations for subsets that seem important after the initial study. For example, if the second subset factors (4,9,10,13,14) appeared inert after the initial 16-runs, we might add a new fraction that reverses all 15 columns except column 4. By reversing columns 9, 10, 13 and 14, but not 4, the second subset reuses the same eight combinations *b1-b8* in the second set of eight runs. This fraction is presented in the lower part of Figure 3. When the two fractions are combined, the result is a $(2_{IV}^{5-1} \times 2_{IV}^{5-1} \times 2_{III}^{5-2})/2^6 = 2_{III}^{15-10}$ design (Design 15-10.3 in [6]). While the design is still resolution III, all three-letter words disappear from the defining relation except the seven words involving factor 4.

What if two-factor interactions for factors within a subset were considered likely? Then the $(2_{IV}^{5-1} \times 2_{IV}^{5-1} \times 2_{IV}^{5-1})/2^7 = 2_{IV}^{15-10}$ design above is less attractive that one with resolution V subsets. Would resolution V 2^{5-1} subsets be possible for a 2_{IV}^{15-10} design? The answer is no, since this is the only 2_{IV}^{15-10} design and it has no length-five words in its defining relation. Thus, to avoid aliasing two-factor interactions within subsets, we look for a $(2_{V}^{5-1} \times 2_{V}^{5-1} \times 2_{V}^{5-1})/2^6 = 2_{IV}^{15-9}$ design. Our first strategy is to consider the 2_{IV}^{15-9} design furnished by software. I requested a 2_{IV}^{15-9} design from JMP (Version 4) and then fit a factorial model up to three-factor interactions - in order to examine the aliasing of two- and three-factor interactions. The aliased effects appeared in JMP's "Singularity Details" output, the first entry being:

X2X6 = X5X9 = X1X13

= X3X7X10 = X4X12X14 = X8X11X15

= X4X8X10 = X3X11X14 = X7X12X15

Note that all 15 factors appear once in the six effects in rows 1 and 2 (or rows 1 and 3). If we pair each two-factor interaction with a three-factor interaction in the second row, we create three subsets of five factors, e.g., (2,6,3,7,10), (5,9,4,12,14), and (1,13,8,11,15). Since each subset equates a two-factor interaction with a three-factor interaction, these subsets are resolution V. If one fits a model with all main effects and the 3×10 two-factor interactions involving factors from within the same subset, the only aliasing is:

X1*X13 = X2*X6 = X5*X9

X13*X15 = X2*X10

X1*X15 = X6*X10

X6*X7 = X4*X5

X2*X7 = X4*X9

X1*X8 = X5*X12

X8*X13 = X9*X12

The other 15 two-factor (within-subset) interactions are clear of aliasing.

Eighteen factor example with resolution IV subsets

The best 2_{III}^{18-13} design has only 16 length-three words in the defining relation. This design is obtained, e.g., if one takes columns 1, 2, 4-7, 8-11, 16-19, and 28-31 (when the columns are arranged in standard order, i.e., 5=1*4, 6=2*4, 7=1*2*4,…, 31=1*2*4*8*16). This design naturally facilitates any of the following designs

- $(2^2 \times 2_{IV}^{4-1} \times 2_{IV}^{4-1} \times 2_{IV}^{4-1} \times 2_{IV}^{4-1})/2^9$ with subsets 1-2, 4-7, 8-11, 16-19, and 28-31
- $(2_{III}^{6-3} \times 2_{IV}^{4-1} \times 2_{IV}^{4-1} \times 2_{IV}^{4-1})/2^7$ with the subset of six factors obtained by combining columns 1-2 with any set of four factors
- $(2^2 \times 2_{IV}^{8-4} \times 2_{IV}^{8-4})/2^5$ obtained by combining sets of four factors
- etc.

The designs with one fewer factor (drop column 2) or one more factor (add column 3) have similar properties. If one drops both columns 1 and 2, the resulting design is a minimal design, i.e., the resolution IV design with k = n/2. Diamond [1, p. 131] describes how minimal designs are particularly well suited for subsets of size 4, 8, 16, etc.

The 2_{III}^{18-13} design above not only accommodates well resolution IV subsets of 4 or 8 factors, it permits a particularly successful follow-up design obtained by reversing a subset of the factors. The best foldover for our 2_{III}^{18-13} is obtained by reversing columns 1-2 and any other set of four columns [8]. This initial design and optimal follow-up fraction are illustrated in Figure 4. Since the subsets are resolution IV, no additional subset treatment combinations are required in the follow-up fraction. This foldover fraction results in a 2_{IV}^{18-12} design providing 18 degrees of freedom for main effects and 43 degrees of freedom for two-factor interactions. If, instead, we used the usual reverse-all-factors foldover for the second fraction, we would have 12 fewer degrees of freedom for two-factor interactions.

FINAL REMARKS

As found in [2], Fisher [9] stated "it is an essential characteristic of experimentation that it is carried out with limited resources, and an essential part of the subject of experimental design to ascertain how those should best be applied...." Using designs that have a minimal number of treatment combinations within subsets of factors addresses this fundamental concern. Bisgaard [2] agrees that in experiments involving the assembly of items or the development of prototypes, manufacturing costs and the cost of experimentation, i.e. testing, are important concerns in the experiment planning process.

This article has focused on the selection of a fractional factorial design that is more economical to conduct because it purposely limits the number of treatment combinations for subsets of factors. When all factors appear in subsets for which only a fraction of the treatment combinations are included, Tables 1 and 2 may be used in selecting a design. If the application you need is not in these tables, we recommend using linear graphs or the alias chains to identify how best to assign your factors to the columns of the design matrix.

Restricted subset designs may be run either as a completely randomized design, as was the case in [2], or they may be run with a split-unit structure [3]. The analysis of the experiment must take this into consideration.

APPENDIX: OTHER COMMENTS REGARDING LINEAR GRAPHS

Any possible restricted subset designs of size 16 can be obtained by deleting factors from one of the Figure 2 graphs. For example, a design for (4,4,6) is easily obtained from (4,4,7) by deleting any one of the factors from the last subset. This is obvious since, by dropping any factor from a 2^{7-4} design, one obtains a 2^{6-3} design. Similarly, by deleting factors, one may obtain a 2^{5-2} design from a 2^{6-3}, and a 2^{3-1} or a (resolution IV) 2^{4-1} from a 2^{5-2} design. Note however, that while the designs so obtained do in fact restrict the number of treatment combinations per subset, they are not necessarily optimal. For example, in reducing one of the Figure 2 graphs down to the nine-factor design (4,5), one may obtain a higher aberration design (designs 9-5.3 or 9-5.5 in [6]) rather than the minimum aberration design 9-5.1 (refer to Table 1). In some cases, we have constructed the Figure 2 linear graphs in a manner than ensures that good designs are obtained when whole subsets are eliminated. For example, retaining any two subsets of size 4 from the (3,4,4,4) graph produces a resolution IV 2^{8-4} design. Other graphs for (3,4,4,4) would not ensure this.

We have provided linear graphs only for 16-run designs. The reader may refer to pp. 1134-1139 of [7] for 32-run design graphs. Resolution III designs with a large number of possible factors are easily recognized from these graphs:

- p. 1134 - graph (2): k = 21: $2^{5-1} \times (2^{4-1})^4 / 2^{11}$
- p. 1137 - graph (8): k = 30: $(2^{6-3})^3 \times (2^{3-1})^4 / 2^{12}$
- p. 1138 - graph (10): k = 30: $2^{6-3} \times (2^{3-1})^8 / 2^{14}$

- p. 1139 - graph (13): k = 28: $(2^{4-1})^7 / 2^{16}$

However, it is difficult to recognize designs for subsets of more than six factors. Also the use of these graphs typically leads to Resolution III designs, even when k is small enough to accommodate a Resolution IV design - unless one purposely selects the columns in the design to achieve a particular design as listed in [6].

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Figure 1. Linear Graphs for Five Small Designs

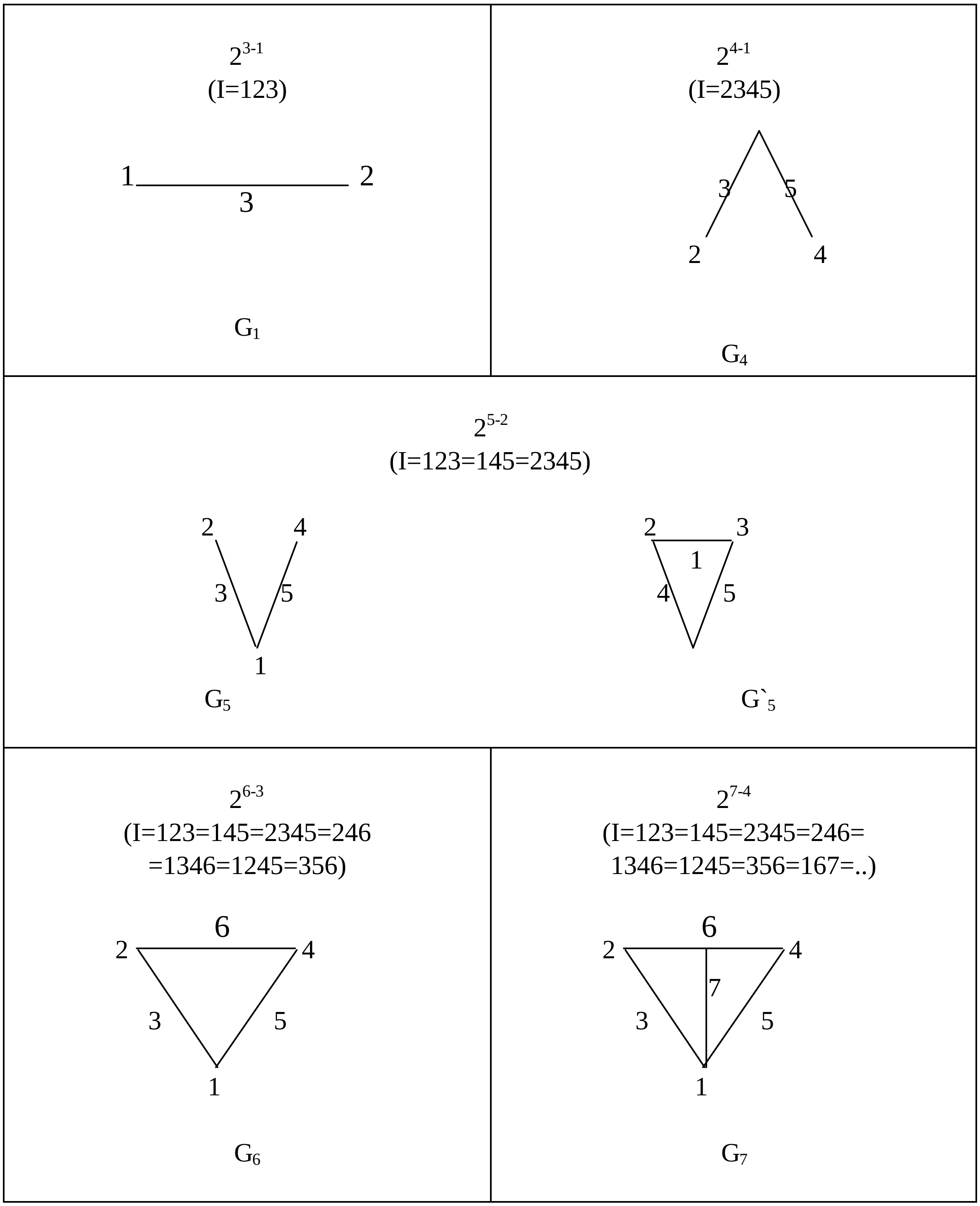

Figure 2. Linear Graphs for Five 2^{15-11} Restricted Subset Designs from Table 1

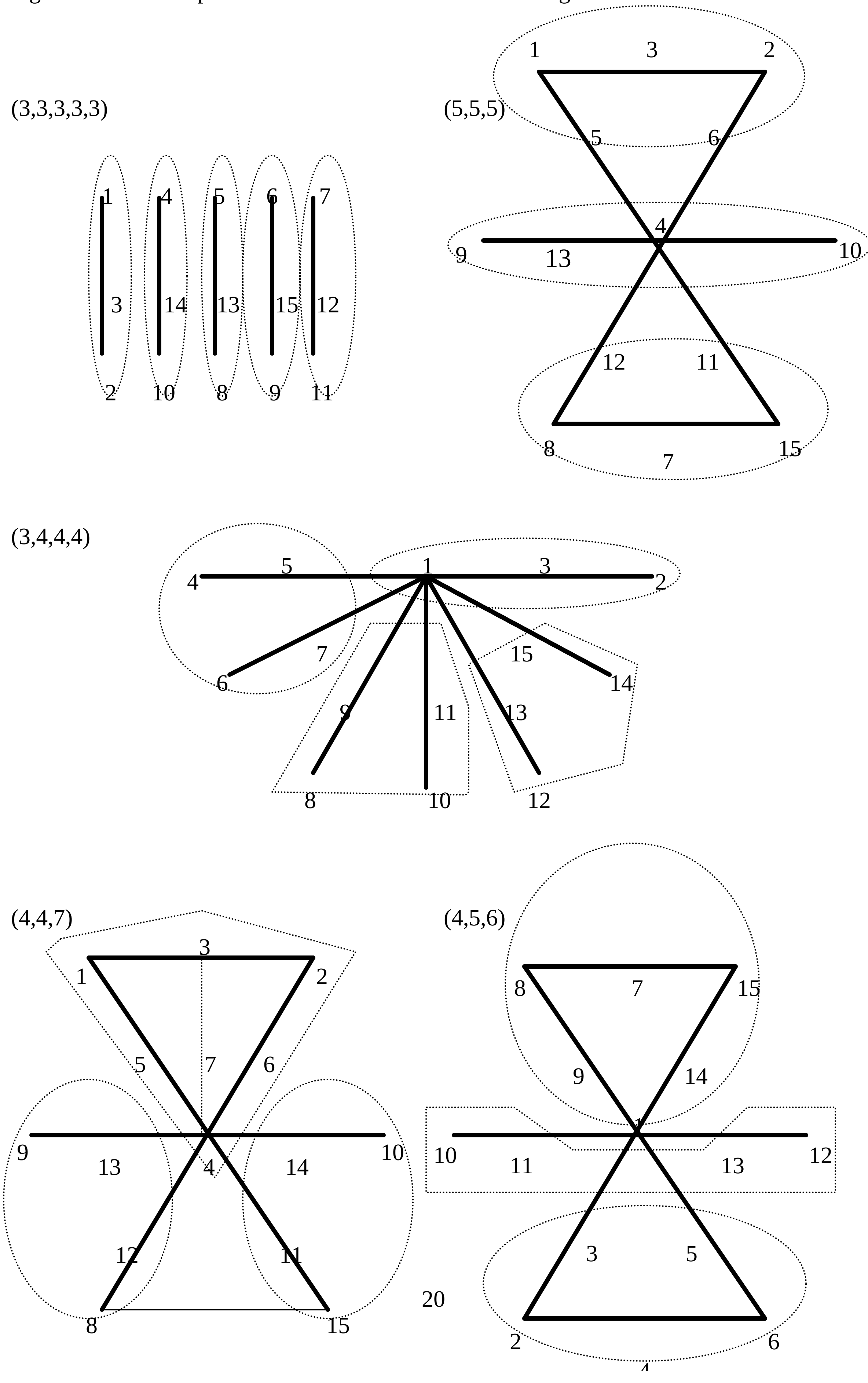

Figure 3. Fifteen factor design with a follow-up design

Initial design is based on assigning columns (3,1,5,2,6) to the five factors in the first subset (denoted A), columns (4,9,10,13,14) to the second (B) subset, and columns (7,8,11,15,12) to the third (C) subset. Each subset has eight treatment combinations, (---++), (--++-), (-+--+), (-++--), (+----), (+-+-+), (++-+-), (+++++), designated 1-8, respectively. For example, *a1* denotes the treatment combination 3=-, 1=-, 5=-, 2=+, 6=+. Each of the eight A subset combinations is used twice, e.g., the first row of the table indicates that subset treatment combination a1 is run with (*b6*, *c3*) and with (*b7*, *c2*).

Initial 2^{15-11}

	b1	*b2*	*b3*	*b4*	*b5*	*b6*	*b7*	*b8*
a1						*c3*	*c2*	
a2		*c7*	*c6*					
a3		*c6*	*c7*					
a4						*c2*	*c3*	
a5					*c8*			*c5*
a6	*c4*			*c1*				
a7	*c1*			*c4*				
a8					*c5*			*c8*

⇓

Foldover 2^{15-11} *obtained by reversing all factors except factor 4*

	b1	*b2*	*b3*	*b4*	*b5*	*b6*	*b7*	*b8*
a1'						*c2'*	*c3'*	
a2'		*c6'*	*c7'*					
a3'		*c7'*	*c6'*					
a4'						*c3'*	*c2'*	
a5'					*c5'*			*c8'*
a6'	*c1'*			*c4'*				
a7'	*c4'*			*c1'*				
a8'					*c8'*			*c5'*

The all-factor foldover of a subset treatment combination is denoted by "' ". For example, *a1*=(-,-,-,+,+), so *a1*' =(+,+,+,-,-). For the *B* subset, we reverse columns 9, 10, 13, and 14, but not column 4. Thus, the foldover of *b1* = (4=-,9=-,10=-,13=+,14=+) is (-,+,+,-,-) = *b4*. Thus, by reversing all but column 4 results in the re-use of the eight *B* subset treatment combinations.

Figure 4. Eighteen Factor Design with a Follow-up Design

Initial 2^{18-13} design with columns assigned to subsets as follows: (1,2), (4-7), (8-11), (16-19), and (28-31). Treatment combinations for subsets are defined as follows: *a1* = (-,-), *a2* = (-, +), *a3* = (+,-), *a4* = (+,+); *b1* = (-,-,-,-), *b2* = (-,-,+,+), *b3* = (-,+,-,+), *b4* = (-,+,+,-), *b5* = (+,-,-,+), *b6* = (+,-,+,-), *b7* = (+,+,-,-), *b8* = (+,+,+,+); define *c1*-*c8*, *d1*-*d8*, and *e1*-*e8* equivalently to *b1*-*b8*. Two subsets of four may be combined to form one subset of eight; e.g., there are 16 combinations for *di* and *ej* (i, j = 1, …, 8).

Initial 2^{18-13}

	b1	*b2*	*b3*	*b4*	*b5*	*b6*	*b7*	*b8*
c1	*a4d1e1* *a4d8e8*							*a4d1e8* *a4d8e1*
c2		*a3d2e2* *a3d7e7*					*a3d2e7* *a3d7e2*	
c3			*a2d3e3* *a2d6e6*			*a2d3e6* *a2d6e3*		
c4				*a1d4e4* *a1d5e5*	*a1d4e5* *a1d5e4*			
c5				*a1d4e5* *a1d5e4*	*a1d4e4* *a1d5e5*			
c6			*a2d3e6* *a2d6e3*			*a2d3e3* *a2d6e6*		
c7		*a3d2e7* *a3d7e2*					*a3d2e2* *a3d7e7*	
c8	*a4d1e8* *a4d8e1*							*a4d1e1* *a4d8e8*

$\Downarrow$

Foldover 2^{18-13} obtained by reversing columns 1,2,4-7

	b1	*b2*	*b3*	*b4*	*b5*	*b6*	*b7*	*b8*
c1	*a1d1e8* *a1d8e1*							*a1d1e1* *a1d8e8*
c2		*a2d2e7* *a2d7e2*					*a2d2e2* *a2d7e7*	
c3			*a3d3e6* *a3d6e3*			*a3d3e3* *a3d6e6*		
c4				*a4d4e5* *a4d5e4*	*a4d4e4* *a4d5e5*			
c5				*a4d4e4* *a4d5e5*	*a4d4e5* *a4d5e4*			
c6			*a3d3e3* *a3d6e6*			*a3d3e6* *a3d6e3*		
c7		*a2d2e2* *a2d7e7*					*a2d2e7* *a2d7e2*	
c8	*a1d1e1* *a1d8e8*							*a1d1e8* *a1d8e1*

In this follow-up fraction, the 32 original (c,d,e) combinations are repeated, each with the mirror image of the original fraction (a,b) combination.

Table 1. Sixteen run designs

Number of Factors and Factors per Subset	Pre-/Post-Fractionation notation	Factor/Column Assignments	[6] Design ID
6 - (3,3)	$(2^{3-1} \times 2^{3-1})/2^0$	$(1,2,3)_4$ $(4,8,12)_4$	6-2.3
7 - (3,4)	$(2^{3-1} \times 2^{4-1})/2^1$	$(1,2,3)_4$ $(4,8,5,9)_8$	7-3.4
8 - (3,5)	$(2^{3-1} \times 2^{5-2})/2^1$	$(1,8,9)_4$ $(2,3,4,5,6)_8$	8-4.5
8 - (4,4)	$(2^{4-1} \times 2^{4-1})/2^2$	$(1,2,4,7)_8$ $(8,11,13,14)_8$	8-4.1
9 - (3,6)	$(2^{3-1} \times 2^{6-3})/2^1$	$(1,8,9)_4$ $(2,3,4,5,6,7)_8$	9-5.5
9 - (4,5)	$(2^{4-1} \times 2^{5-2})/2^1$	$(8,9,14,15)_8$ $(1,2,3,4,5)_8$	9-5.1
9 - (3,3,3)	$(2^{3-1} \times 2^{3-1} \times 2^{3-1})/2^2$	$(1,2,3)_4$ $(4,8,12)_4$ $(5,10,15)_4$	9-5.2
10 - (4,6)	$(2^{4-1} \times 2^{6-3})/2^2$	$(2,3,4,5)_8$ $(1,6,8,9,14,15)_8$	10-6.1
10 - (5,5)	$(2^{5-2} \times 2^{5-2})/2^2$	$(1,2,3,4,5)_8$ $(6,8,9,14,15)_8$	10-6.1
10 - (3,3,4)	$(2^{3-1} \times 2^{3-1} \times 2^{4-1})/2^3$	$(1,2,3)_4$ $(5,6,9,10)_8$ $(4,8,12)_4$	10-6.3
10 - (3,7)		Does Not Exist	
11 - (4,7)	$(2^{4-1} \times 2^{7-4})/2^2$	$(8,9,10,11)_8$ $(1,2,3,4,5,6,7)_8$	11-7.3
11 - (5,6)	$(2^{5-2} \times 2^{6-3})/2^2$	$(3,8,9,10,11)_8$ $(1,2,4,5,6,7)_8$	11-7.3
11 - (3,4,4)	$(2^{3-1} \times 2^{4-1} \times 2^{4-1})/2^4$	$(1,2,3)_4$ $(4,6,10,14)_8$ $(5,8,9,13)_8$	11-7.1
11 - (3,3,5)	$(2^{3-1} \times 2^{3-1} \times 2^{5-2})/2^3$	$(1,8,9)_4$ $(6,10,12)_4$ $(2,3,4,5,7)_8$	11-7.2
12 - (3,4,5)	$(2^{3-1} \times 2^{4-1} \times 2^{5-2})/2^4$	$(4,8,12)_4$ $(2,5,6,7)_8$ $(1,3,9,10,11)_8$	12-8.2
12 - (4,4,4)	$(2^{4-1} \times 2^{4-1} \times 2^{4-1})/2^5$	$(2,3,4,5)_8$ $(1,6,10,13)_8$ $(8,9,14,15)_8$	12-8.1
12 - (3,3,3,3)	$(2^{3-1} \times 2^{3-1} \times 2^{3-1} \times 2^{3-1})/2^4$	$(1,2,3)_4$ $(4,10,14)_4$ $(5,8,13)_4$ $(6,9,15)_4$	12-8.1
12 - (5,7) (6,6) (3,3,6)		Do Not Exist	
13 - (3,4,6)	$(2^{3-1} \times 2^{4-1} \times 2^{6-3})/2^4$	$(1,6,7)_4$ $(8,10,12,14)_8$ $(2,4,9,11,13,15)_8$	
13 - (3,5,5)	$(2^{3-1} \times 2^{5-2} \times 2^{5-2})/2^4$	$(1,6,7)_4$ $(2,3,8,9,10)_8$ $(4,5,11,14,15)_8$	
13 - (4,4,5)	$(2^{4-1} \times 2^{4-1} \times 2^{5-2})/2^5$	$(8,9,14,15)_8$ $(10,11,12,13)_8$ $(1,2,4,6,7)_8$	
13 - (6,7) (3,3,7) (3,3,3,4)		Do Not Exist	

14 - (4,4,6)	$(2^{4-1} \times 2^{4-1} \times 2^{6-3})/2^5$	$(8,11,12,15)_8$ $(9,10,13,14)_8$ $(1,2,4,5,6,7)_8$
14 - (4,5,5)	$(2^{4-1} \times 2^{5-2} \times 2^{5-2})/2^5$	$(6,7,12,13)_8$ $(2,3,8,9,10)_8$ $(4,5,11,14,15)_8$
14 - (7,7) (3,4,7) (3,5,6) (3,3,3,5) (3,3,4,4)		Do Not Exist
15 - (4,4,7)	$(2^{4-1} \times 2^{4-1} \times 2^{7-4})/2^5$	$(8,9,12,13)_8$ $(10,11,14,15)_8$ $(1,2,3,4,5,6,7)_8$
15 - (4,5,6)	$(2^{4-1} \times 2^{5-2} \times 2^{6-3})/2^5$	$(10,11,12,13)_8$ $(2,3,4,5,6)_8$ $(1,7,8,9,14,15)_8$
15 - (5,5,5)	$(2^{5-2} \times 2^{5-2} \times 2^{5-2})/2^5$	$(1,2,3,5,6)_8$ $(4,9,10,13,14)_8$ $(7,8,11,12,15)_8$
15 - (3,4,4,4)	$(2^{3-1} \times 2^{4-1} \times 2^{4-1} \times 2^{4-1})/2^5$	$(1,2,3)_4$ $(4,5,6,7)_8$ $(8,9,10,11)_8$ $(12,13,14,15)_8$
15 - (3,3,3,3,3)	$(2^{3-1} \times 2^{3-1} \times 2^{3-1} \times 2^{3-1} \times 2^{3-1})/2^6$	$(1,2,3)_4$ $(4,10,14)_4$ $(5,8,13)_4$ $(6,9,15)_4$ $(7,11,12)_4$
15 - (3,5,7) (3,6,6) (3,3,3,6) (3,3,4,5)		Do Not Exist

The first column lists the total number of factors studied as well as the number of factors per subset. The second column represents the design using notation from [2]. Column 3 indicates how to assign the factors to columns in Table 3 to construct the design; the factors' column assignments are in parenthesis with the number of treatment combinations per subset indicated by a subscript. The last column identifies the design as in [6]. Although resolution IV designs exist for k = 6, 7, and 8, to restrict the number of combinations for a subset of three factors, we must use Resolution III designs. The only exception is the (4,4) design for k=8. For k = 13, 14, and 15, the resolution III design is unique.

Table 2. Thirty-two run, resolution IV designs

Number of Factors and Factors per Subset	Pre-/Post-Fractionation notation	Factor/Column Assignments	[6] Design ID
8 - (4,4)	$(2^{4-1} \times 2^{4-1})/2^1$	$(1,2,4,7)_8$ $(8,11,16,19)_8$	8-3.3
9 - (4,5)	$(2^{4-1} \times 2^{5-1})/2^2$	$(1,2,4,7)_8$ $(8,11,16,19,29)_{16}$	9-4.1
10 - (4,6)	$(2^{4-1} \times 2^{6-2})/2^2$	$(1,2,4,7)_8$ $(8,11,16,19,29,30)_{16}$	10-5.1
10 - (5,5)	$(2^{5-1} \times 2^{5-1})/2^3$	$(1,2,4,7,8)_{16}$ $(11,16,19,29,30)_{16}$	10-5.1
11 - (4,7)	$(2^{4-1} \times 2^{7-3})/2^2$	$(1,2,4,7)_8$ $(8,11,13,14,16,19,21)_{16}$	11-6.2
11 - (5,6)	$(2^{5-1} \times 2^{6-2})/2^3$	$(1,7,11,13,16)_{16}$ $(2,4,8,19,21,25)_{16}$	11-6.1
12 - (4,8)	$(2^{4-1} \times 2^{8-4})/2^2$	$(16, 19,21,22)_8$ $(1,2,4,7,8,11,13,14)_{16}$	12-7.2
12 - (5,7)	$(2^{5-1} \times 2^{7-3})/2^3$	$(8,11,13,14,25)_{16}$ $(1,2,4,7,16,19,21)_{16}$	12-7.1
12 - (6,6)	$(2^{6-2} \times 2^{6-2})/2^3$	$(1,2,13,14,21,25)_{16}$ $(4,7,8,11,16,19)_{16}$	12-7.1
12 - (4,4,4)	$(2^{4-1} \times 2^{4-1} \times 2^{4-1})/2^4$	$(1,2,4,7)_8$ $(8,11,13,14)_8$ $(16,19,21,22)_8$	12-7.2
13 - (5,8)	$(2^{5-1} \times 2^{8-4})/2^3$	$(1,2,4,7,25)_{16}$ $(8,11,13,14,16,19,21,22)_{16}$	13-8.1
13 - (6,7)	$(2^{6-2} \times 2^{7-3})/2^3$	$(4,7,8,11,16,19)_{16}$ $(1,2,13,14,21,22,25)_{16}$	13-8.1
13 - (4,4,5)	$(2^{4-1} \times 2^{4-1} \times 2^{5-1})/2^5$	$(1,2,4,7)_8$ $(8,11,13,14)_8$ $(16,19,21,22,25)_{16}$	13-8.1
14 - (6,8)	$(2^{6-2} \times 2^{8-4})/2^3$	$(4,7,8,11,16,19)_{16}(1,2,13,14,21,22,25,26)_{16}$	14-9.1
14 - (7,7)	$(2^{7-3} \times 2^{7-3})/2^3$	$(1,4,8,13,16,21,25)_{16}$ $(2,7,11,14,19,22,26)_{16}$	14-9.1
14 - (4,5,5)	$(2^{4-1} \times 2^{5-1} \times 2^{5-1})/2^6$	$(1,2,4,7)_8$ $(8,11,13,14,16)_{16}$ $(19,21,22,25,26)_{16}$	14-9.1
14 - (4,4,6)	$(2^{4-1} \times 2^{4-1} \times 2^{6-2})/2^5$	$(1,2,4,7)_8$ $(8,11,13,14)_8$ $(16,19,21,22,25,26)_{16}$	14-9.1
15 - (7,8)	$(2^{7-3} \times 2^{8-4})/2^3$	$(1,7,11,13,19,21,25)_{16}$ $(2,4,8,14,16,22,26,28)_{16}$	15-10.1
15 - (4,4,7)	$(2^{4-1} \times 2^{4-1} \times 2^{7-3})/2^5$	$(2,4,8,14)_8$ $(16,22,26,28)_8$ $(1,7,11,13,19,21,25)_{16}$	15-10.1
15 - (4,5,6)	$(2^{4-1} \times 2^{5-1} \times 2^{6-2})/2^6$	$(1,2,4,7)_8$ $(8,11,13,14,16)_{16}$ $(19,21,22,25,26,28)_{16}$	15-10.1
15 - (5,5,5)	$(2^{5-1} \times 2^{5-1} \times 2^{5-1})/2^7$	$(1,2,4,7,28)_{16}$ $(8,11,13,14,16)_{16}$ $(19,21,22,25,26)_{16}$	15-10.1
16 - (8,8)	$(2^{8-4} \times 2^{8-4})/2^3$	$(1,2,4,7,8,11,13,14)_{16}$ $(16,19,21,22,25,26,28,31)_{16}$	16-11.1
16 - (4,6,6)	$(2^{4-1} \times 2^{6-2} \times 2^{6-2})/2^5$	$(1,2,4,7)_8$ $(8,11,13,14,19,21)_{16}$ $(16,22,25,26,28,31)_{16}$	16-11.1
16 - (4,4,8)	$(2^{4-1} \times 2^{4-1} \times 2^{8-4})/2^5$	$(1,2,4,7)_8$ $(8,11,13,14)_8$ $(16,19,21,22,25,26,28,31)_{16}$	16-11.1

16 - (4,5,7)	$(2^{4-1} \times 2^{5-1} \times 2^{7-3})/2^6$	$(1,2,4,7)_8$ $(8,11,13,14,16)_{16}$ $(19,21,22,25,26,28,31)_{16}$	16-11.1
16 - (5,5,6)	$(2^{5-1} \times 2^{5-1} \times 2^{6-2})/2^7$	$(1,2,4,7,19)_{16}$ $(8,11,13,14,16)_{16}$ $(21,22,25,26,28,31)_{16}$	16-11.1
16 - (4,4,4,4)	$(2^{4-1} \times 2^{4-1} \times 2^{4-1} \times 2^{4-1})/2^7$	$(1,2,4,7)_8$ $(8,11,13,14)_8$ $(16,19,21,22)_8$ $(25,26,28,31)_8$	16-11.1

The first column lists the total number of factors studied as well as the number of factors per subset. The second column represents the design using notation from [2]. Column 3 indicates how to assign the factors to columns in Table 3 to construct the design; the factors' column assignments are in parenthesis with the number of treatment combinations per subset indicated by a subscript. The last column identifies the design as in [6]. For k = 13, 14, 15 and 16, the resolution IV design is unique.

Table 3. The design matrix for 16 and 32 run designs. The independent columns are numbered 1, 2, 4, 8, and 16. The first 4 rows and 15 columns are used for 16-run designs, and all 5 rows and 31 columns are used for 32-run designs. The column set is in the standard Yates order.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
17	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	

Table 4. A Sixteen-Run, Resolution IV Design for 10 Two-Level Factors

Run	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10
1	-1	-1	-1	-1	1	-1	-1	1	-1	-1
2	-1	-1	-1	1	-1	1	1	-1	1	-1
3	-1	-1	1	-1	-1	1	1	-1	-1	1
4	-1	-1	1	1	1	-1	-1	1	1	1
5	-1	1	-1	-1	-1	1	-1	1	1	1
6	-1	1	-1	1	1	-1	1	-1	-1	1
7	-1	1	1	-1	1	-1	1	-1	1	-1
8	-1	1	1	1	-1	1	-1	1	-1	-1
9	1	-1	-1	-1	-1	-1	1	1	1	1
10	1	-1	-1	1	1	1	-1	-1	-1	1
11	1	-1	1	-1	1	1	-1	-1	1	-1
12	1	-1	1	1	-1	-1	1	1	-1	-1
13	1	1	-1	-1	1	1	1	1	-1	-1
14	1	1	-1	1	-1	-1	-1	-1	1	-1
15	1	1	1	-1	-1	-1	-1	-1	-1	1
16	1	1	1	1	1	1	1	1	1	1